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Deriving a common set of weights: A Double frontier Approach

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Abstract: Data Envelopment Analysis (DEA) as a non-parametric method for efficiency measurement allows decision making units (DMUs) to select the most favorable weight in order to maximize their efficiency scores. This evaluation determines the best efficiency score. Generally, the performance of DMUs can also be evaluated from pessimistic perspective. As a result, the performance of different units is achieved with two different evaluations, namely optimistic and pessimistic. On the other hand, different set of weights assigns different performance for each unit. So, common set of weights make a basis for comparison and ranking units. This paper develops the weight restriction approach to integrate both efficiencies in the form of an interval. The proposed weight restriction method not only integrate both efficiency scores but also generates positive and dissimilar set of weights. Based on the common set of weights, the efficiency scores are calculated then the units are ranked. To elucidate the details of the proposed method, a real world data set that characterizes the performance of seventeen forest districts demonstrates the practicality and superiority of the proposed method.

Keywords: Data Envelopment Analysis, Common set of weights, Weight restriction, optimistic and pessimistic efficiencies, and weight dissimilarity.

1. Introduction

Data envelopment analysis (DEA) is concerned with comparative assessment of the efficiency of decision making units (DMUs*)*. In classical DEA models, the efficiency of a DMU is obtained by maximizing the ratio of the weighted sum of its outputs to the weighted sum of its inputs, subject to the condition that this ratio does not exceed one for any DMU. Since the pioneering work of Charnes et.al [6] and Banker et.al [4] DEA has demonstrated to be an effective technique for measuring the relative efficiency of a set of homogeneous DMUs in many contexts. Specifically, the flexibility of standard DEA models in choosing a set of weights for inputs and outputs, often causes more than one DMU being evaluated as efficient. What's more, leading them being unable to be fully discriminated. One of the possible ways for solving this problem lies in the specification of a common set of weights (CWS). Many approaches have been proposed to achieve a common set of weights. For example refer to Pourhabib et.al [11], Ramon et.al [12], Roll et.al [13], Wu et.al [16], Eyni et.al [8] and some other researchers. In before mentioned papers, the proposed method can be known as a method for analysis the best relative efficiency or optimistic efficiency. In their proposed model, a DMU is specified as DEA efficient or optimistic efficient if its best relative efficiency equals

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one; otherwise, it is called DEA-non-efficient or optimistic non-efficient. Placing emphasis on the non-performing units, the performance of units can be evaluated from the pessimistic point of view. The worst relative efficiency or pessimistic point of view assigns the most unfavorable weights to each unit. If the optimal value of the model is equal to unity, that DMU is called as DEA-inefficient or pessimistic inefficient; otherwise, it is said to be DEA-pessimistic non efficient. In order to have a general scenario about the performance of a DMU, applying both points of view, optimistic and pessimistic is practically more useful. To over hatching the benefits of both perspectives in practice, Azizi [1] presented a bounded model for obtaining an interval efficiency using the concept of optimistic and pessimistic efficiencies. The author also highlighted the shortcoming of Entani's model, namely, Entani's model [7] does not take all input and outputs in the evaluation, and so, it is not able to identify an adequate bound for interval efficiencies. Azizi et.al [2] pointed out to the drawback of existing model for evaluating interval efficiency and proposed pair of revised models that make it possible to perform a DEA efficiency analysis based on the new interval efficiency models. Salahi et.al [14] suggested an equivalent formulation of the robust envelopment CCR model in presence of input and output uncertainty. What's more, the authors proposed a linear programming for deriving a common set of weights (CSW) under uncertainty. Arabmaldar et.al [3] proposed an approach for handling uncertainty in presence of interval data. A key advantage of this approach is focusing on the worst performing frontier with non-discretionary factors. Using overall performance measures, Jahed et.al [9] proposed an overall performance measures for evaluating DMUs developing the fuzzy DEA theory and methodology. The authors proposed a fuzzy DEA models that evaluate a DMU from the pessimistic perspective in a fuzzy context. Finally, using the double frontier analysis approach, a measure for evaluating the performance is obtained. Tapia et.al [15] focused on the measuring efficiency problem as a statistical problem. The authors proposed two confidence interval methodologies. One is inspired in the optimistic/pessimistic point of view of DEA models and the other in the use of bootstrap replications from the sample of customers in each DMU. Readers may consult some recent references for further results. Optimistic and pessimistic efficiencies measure two extremes of each DMU performance. To determine the overall performance of each DMU, both perspectives should be considered simultaneously. An approach that evaluates the performance of each DMU for both optimistic and pessimistic efficiencies is called "double frontier analysis" approach. However, sometimes the researchers have made some contributions to deal with the common set of weights (CSW) employing double frontier analysis. Since, applying one of the efficiencies suffers from bias. In this paper, we aim to search one common set of weights employing the interval efficiency of each DMU and then rank the DMUs with these obtained interval efficiency scores. The proposed weight restriction approach generates positive weights and, at the same time, prevents weights dissimilarity when an interval efficiency is taken into account. The rest of the paper has the following order. The next section will present the basic DEA method for measuring interval efficiencies and weight restriction approach in DEA literature. In the section to follow a common set of weights (CSW) is found with employing an interval efficiency along with weight restriction approach. Numerical examples are discussed in section4, and conclusions are offered in section section5

2. Literature Review

Since the performances of DMUs can be measured from both optimistic and pessimistic views, two efficiencies are obtained for each DMU: optimistic and pessimistic efficiency. Consider a set of DMU indexed by J . For all $j \in J = \{1,\ldots,n\}$, DMU_j uses input $x_{ij}(i=1,\ldots,m)$ to produce $y_{rj}(r=1,\ldots,s)$. Also, for each $j \in J$, the input and output value of DMU_j are known and positive. The following multiplier form of CCR presented by Charnes et.al [6] measures the best relative efficiency of DMU_j :

$$
\begin{aligned}
\max \quad & \theta_o = \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}} \\
& \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1 \quad , j=1,\ldots,n \qquad (1) \\
& u_r, v_i \ge \varepsilon \; , r=1,\ldots,s, i=1,\ldots,m
\end{aligned}
$$

In the above model, $u_r(r=1,\ldots,s)$ and $v_i(i=1,\ldots,m)$ denote the weight value for $r-th$ output and $i-th$ input respectively, and ε is a non-Archimedean infinitesimal number. Employing Charnes and Cooper [5] transformation, the above model is converted to linear programming model as follows:

$$
\begin{aligned}
\max \quad & \theta_o = \sum_{r=1}^{s} u_r y_{ro} \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \quad , j=1,\ldots,n \\
& \sum_{i=1}^{m} v_i x_{io} = 1 \qquad (2) \\
& u_r, v_i \ge \varepsilon \; , r=1,\ldots,s \; , i=1,\ldots,m
\end{aligned}
$$

The above linear model (2) measures the best relative efficiency of DMUs in the output-oriented mode. If the optimal value of the objective function in model (2) is one, $\theta_o^* = 1$ DMU_o is said to be DEA-efficient or optimistic efficient; otherwise, it is DEA-non-efficient or optimistic non-efficient. From the pessimistic view, the worst efficiency score are evaluated relative to DMUs on the worst performing frontier. The following model is expressed as pessimistic DEA model:

$$\min \quad \varphi_o = \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}}$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \geq 1 \quad , j = 1,\ldots,n \quad (3)$$

$$u_r, v_i \geq \varepsilon \quad r=1,\ldots,s, i=1,\ldots,m$$

Rearranging the model (3) by Charnes and Cooper [5] transformation, the problem (3) can be converted into a linear program:

$$\min \quad \varphi_o = \sum_{r=1}^{s} u_r y_{ro}$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \geq 0 \quad , j = 1,\ldots,n$$

$$\sum_{i=1}^{m} v_i x_{io} = 1 \quad (4)$$

$$u_r, v_i \geq \varepsilon , r = 1,\ldots,s, i = 1,\ldots,m$$

The model (4) identifies the worst performing unit by assigning the most unfavorable weights to each DMU in the unfavorable scenario. If in optimality, $\varphi_o^* = 1$ DMU_o is said to be DEA-inefficient or pessimistic inefficient.

Research Findings

Theoretically, the best and the worst relative efficiencies should form an interval. For this purpose, the pessimistic efficiency should be adjusted. Assume that $\alpha (0 < \alpha \leq 1)$ is the adjustment factor. The adjusted interval efficiency can be written as $[\alpha_j \varphi_j^*, \theta_j^*] (j = 1,\ldots,n)$ so that the condition $\alpha_j \varphi_j^* \leq \theta_j^*, (j = 1,\ldots,n)$ holds for all intervals $[\alpha_j \varphi_j^*, \theta_j^*] (j = 1,\ldots,n)$. Pinning with this parameter $\alpha_j (j = 1,\ldots,n)$, in order to search a positive lower bound for common set of weights among all feasible multipliers, a joint weight restriction approach [3] is applied. The joint weight restriction approach proposed by Pourhabib et.al [11] allows selecting common weights through conjointly restricting the input and output weights with a common bound. Again suppose there are n units, and each unit uses input $x_{ij} (i = 1,\ldots,m)$ to produce $y_{rj} (r = 1,\ldots,s)$. Also, for each $j = 1,\ldots,n$. The model proposed by Pourhabib et.al [11] has the following format:

$$\min \frac{\sum_{j=1}^{n} d_j}{\alpha}$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + d_j = 0 \quad , j = 1,\dots,n \quad (5)$$

$$\alpha \le v_i \le 1, \quad i = 1,\dots,m$$

$$\alpha \le u_r \le 1, \quad r = 1,\dots,s$$

In the model (5), the variable $d_j (j = 1,\dots,n)$ is denoted as deviation variable or slack variable for each unit and α shows lower bound for both input and output weight. Moreover, all weights do not exceed the upper bound (which is unity). The objective function minimizes the summation of deviation variable and maximizes the lower bound.

Equipped with this approach, in order to have a common set of weight considering adjusted interval efficiency, namely, $[\alpha_j \varphi_j^*, \theta_j^*] (j = 1,\dots,n)$ the following model can be structured:

$$\min \quad \eta = \frac{\sum_{j=1}^{n} s_{1j} + \sum_{j=1}^{n} s_{2j}}{\delta}$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + s_{1j} = 0 \quad , j = 1,\dots,n \quad (6)$$

$$\sum_{r=1}^{s} u_r y_{rj} - (\alpha_j) \sum_{i=1}^{m} v_i x_{ij} - s_{2j} = 0$$

$$\alpha_j \varphi_j^* \le \theta_j^* \quad j = 1,\dots,n$$

$$\delta \le v_i \le 1, \quad i = 1,\dots,m$$

$$\delta \le u_r \le 1, \quad r = 1,\dots,s$$

It is clear that the above model (6) is nonlinear. The objective function minimizes the summation of deviation variable and maximizes the lower bound of weights. As the third constraint claims θ_j^* and φ_j^* are the upper and lower bound of interval efficiency, respectively. They actually form an interval efficiency $[\varphi_j^*, \theta_j^*] (j = 1,\dots,n)$. Theoretically the lower bound of the interval should be adjusted, the variable $\alpha_j (j = 1,\dots,n)$ holds the condition that $\alpha_j \varphi_j^* \le \theta_j^*, (j = 1,\dots,n)$. Therefore, employing the best and the worst relative efficiencies, the proposed weight restriction approach generates positive weights and prevents weight dissimilarity. In a nutshell, model (6) finds a common set of weights using the best and worst efficiency scores. The positive and non-zero weights are also applied for adequate ranking of DMUs. The following theorem proves that the proposed weight restriction approach is feasible and generates positive weights.

Theorem1: Model (6) is always feasible and generates positive weights in optimality.
Proof: refer to Pourhabib et.al [11].

3. Case Study

The applicability of the proposed approach is illustrated by an empirical data set consisting of seventeen forest district from Kao and Hung [10]. Four inputs including Budget in US dollars ($I1$), initial stocking in cubic meters ($I2$), labor in number of employees ($I3$) and land in hectares ($I4$) are used to produce three outputs, namely, main product in cubic meters ($O1$), soil conversation in cubic meters ($O2$) and recreation in number of visits ($O3$). Table1 shows Data set.

Table1: Data Set of seventeen forest districts

DMU	$I1$	$I2$	$I3$	$I4$	$O1$	$O2$	$O3$
DMU1	51.62	11.23	49.22	33.52	40.49	14.89	3166.71
DMU2	85.78	123.98	55.13	108.46	43.51	173.93	6.45
DMU3	66.65	104.18	257.09	13.65	139.74	115.96	0
DMU4	27.87	107.6	14	146.43	25.47	131.79	0
DMU5	51.28	117.51	32.07	84.5	46.2	144.99	0
DMU6	36.05	193.32	59.52	8.23	46.88	190.99	822.29
DMU7	25.83	105.8	9.51	227.2	19.4	120.09	0
DMU8	123.02	82.44	87.35	98.8	43.33	125.84	404.69
DMU9	61.95	99.77	33	86.37	45.43	79.6	1252.6
DMU10	80.33	104.65	53.3	79.06	27.28	132.49	42.76
DMU11	205.92	183.49	144.16	59.66	14.09	196.29	16.15
DMU12	82.09	104.94	46.51	127.25	44.87	108.53	0
DMU13	202.21	187.74	149.39	93.65	44.97	184.77	0
DMU14	67.55	82.83	44.37	60.85	26.04	85	23.95
DMU15	72.6	132.73	44.46	173.48	5.55	135.65	24.13
DMU16	84.83	104.28	159.12	171.11	11.53	110.22	49.09
DMU17	71.77	88.16	69.19	123.14	44.83	74.54	6.14

Models (2) and (4) are performed on the data set of Table1. The results of optimistic and pessimistic efficiencies are recorded in Table 2.

Table2: The Results of Models (2) and (4)

DMU	θ_o^*	φ_o^*
DMU1	1	1
DMU2	1	1.96
DMU3	1	1
DMU4	1	1.14
DMU5	0.95	1.24
DMU6	1	1.07
DMU7	1	1
DMU8	0.78	1.10
DMU9	0.90	1
DMU10	0.65	1.30

DMU11	0.74	1
DMU12	0.47	1
DMU13	0.52	1
DMU14	0.59	1.09
DMU15	0.53	1
DMU16	0.48	1
DMU17	0.42	1
Average	0.76	1.11
variance	0.05	0.05

Equipped with these efficiencies, model (6) are performed on the data set of Table1.the common set of weights generated by model (6) are presented in Table3.

Table3: Common set of weights generated by model (6)

	INPUT				OUTPUT		
	$I1$	$I2$	$I3$	$I4$	$O1$	$O2$	$O3$
Common weights	0.01	0.01	0.03	1	0.18	0.01	1

Table4 represents the efficiency score of these seventeen forest districts employing the common set of weights recorded in Table3.

Table4: The result of Common weights for DMUs

DMU	Efficiency with common weights	Rank	θ_o^* Optimistic Efficiency Model(2)	Rank
DMU1	0.96327	1	1	1
DMU2	0.096286	8	1	1
DMU3	0.263128	3	1	1
DMU4	0.059025	12	1	1
DMU5	0.097659	7	0.95	2
DMU6	0.771506	2	1	1
DMU7	0.046929	15	1	1
DMU8	0.129688	5	0.78	4
DMU9	0.230512	4	0.90	3
DMU10	0.067535	11	0.65	6
DMU11	0.04737	14	0.74	5
DMU12	0.091619	9	0.47	11
DMU13	0.099423	6	0.52	9
DMU14	0.059133	13	0.59	7
DMU15	0.024919	16	0.53	8
DMU16	0.034537	17	0.48	10
DMU17	0.088632	10	0.42	12
Average	0.19	---	0.76	
Variance	0.07	---	0.05	

Regarding to Table4, efficiency scores calculated by the obtained common set of weights are recorded in the second column of Table4. It can be seen that the proposed weight restriction approach along with the adjusted interval efficiency has more discrimination on DMUs. From the statistical point of view, reported in the last row of Table4, the proposed approach attains the least value, 0.19. While, the average score of efficiency is 0.76 in optimistic evaluation. The results imply that the variance value in the proposed common set of weights is about 0.07 which is larger than the optimistic evaluation. Thus, the proposed weight restriction approach not only leads to strictly positive weights but also prevents dissimilar weights.

4. Conclusion

Standard DEA models suffers flexibility in selecting inputs / output weights for calculating the efficiency scores. On the other hand, conventional form of DEA models evaluates DMUs from the optimistic point of view. In order to obtain an overall assessment of the performance of each DMU, we need to integrate different performance measures. In other words, the evaluation from pessimistic point of view or the worst performing frontier. A performance score should consist of both evaluations, pessimistic and optimistic perspective. Equipped with both evaluations, this paper employs a joint weight restriction approach to generate a common set of weights for all DMUs. A key advantage of this approach is focusing on both evaluation to identify positive and dissimilar weights for inputs and outputs. The practical application of this methodology for

Evaluating seventeen forest districts illustrated the strength of developed weight restriction approach in generating positive and dissimilar weights.

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