An approximate method for solving fractional partial differential equation by using an embedding process

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In this article, solving of fractional partial differential equation (FPDE) with optimal time control approach is developed. Conformable derivative as a new definition of fractional derivative is considered. At first the (FPDE) with stationary regime is converted to optimal time control problem, then by using an embedding process, the obtained system is converted to finite-dimensional linear programming (LP) problem and finally optimal time corresponded by (FPDE) system is approximated. For example, this method is used on the conformable fractional heat equation with the initial and boundary conditions.

keywords: Fractional partial differential equation, Optimal control problem, Embedding process, Conformable fractional derivative, Linear programming.

1 Introduction

Partial differential equation (PDE) as an interest issue for modelling of natural phenomenal such as chemical physics, fluid flows and viscoelasticity is studied by many authors. ([1], [2], [3]). By extra information in fractional calculus a lot of abnormal physical systems such as fractional heat equation, fractional wave equation is described by (FPDE)'s. In many works, Riemann-Liouville, Caputo and Grünwald-Letnikov fractional operators is applied to modelling and interpreting of system behaviors ([4], [5]). In this work we present a novel approach to solving FPDE's by using Conformable fractional derivative(CFD), that is introduced by khalil ([6]).(CFD) as a local and limit -based definition is expanded rapidly and used in many applications. however in this work all major initial and boundary conditions on (FPDE) is considered and embedding process, as a useful method is examined to solve this problems. As an advantages of the proposed method can be mentioned that, it is self-starting and not iterative, so this method has been developed to solve a variety of (FPDE) problems. Authors used (CFD) to solving Fractional optimal control problems successfully in previous works such as ([8]) and now are going to apply this definition for modelling a fractional heat equation in an unknown minimum time as an example of FPDE's. Using moment form and an embedding process respectively, problem can be approximated by finite-dimensional (LP) one's.

2 Preliminaries

In this section some basic concepts of conformable fractional derivative is mentioned. We assume $w = f(t)$ $(t \geq 0)$ be a real valued and continuous function, and $\alpha \geq 0$ is a given real number.

Definition 2.1 Let $f:[0,\infty] \to R$, then the (CFD) of $f(t)$ is defined as follows:([6])

$$T_\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon} \quad , \qquad 0 \le \alpha < 1, t \ge 0. \tag{1}$$

We write sometimes $f^{(\alpha)}(t)$ for $T_\alpha f(t)$ to denote (CFD) of order α, also if $T_\alpha f(t)$ exists, then we say f is α-differentiable.

Let $\alpha \in (0,1]$ and f, g be α-differentiable for $t > 0$, then the following properties can be resulted by the (CFD):

$$T_\alpha(af+bg) = aT_\alpha(f) + bT_\alpha(g), \ \ a, b \in R, \tag{2}$$

$$T_\alpha(t^p) = pt^{p-\alpha}, \qquad p \in R, \tag{3}$$

$$T_\alpha(\lambda) = 0, \qquad \lambda = const, \tag{4}$$

$$T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f), \tag{5}$$

$$T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}. \tag{6}$$

Moreover, if f be a differentiable function, then one can prove that

$$T_\alpha f(t) = t^{1-\alpha} \frac{df(t)}{dt}. \tag{7}$$

Definition 2.2 Let $f:(0,t) \to R$ $(t \ge 0)$, be a continuous function and $\alpha \in (0,1)$, the conformable α-fractional integral of a function f is defined as:

$$I_\alpha f(t) = \int_0^t \tau^{\alpha-1} f(\tau) d\tau. \tag{8}$$

Theorem 2.3 Let $f:[0,\infty] \to R$ be a function such that f is differentiable and also α-differentiable. Let g be a function defined in the range of f and also differentiable, then we have the following rule:

$$T_\alpha(fog)(t) = (T_\alpha f(g(t)).(T_\alpha g(t)).g(t)^{\alpha-1}. \tag{9}$$

Theorem 2.4 Let f be a differentiable function for $t > 0$, and $0 < \alpha \le 1$, then

$$I_\alpha T_\alpha f(t) = f(t) - f(0), \tag{10}$$

3 System description and formulation

Consider the following time fractional partial heat equation in conformable difinition sense: In the region $\Pi = [0,1] \times [0,T]$ and $0 < \alpha \le 1$, the function $f(x,t)$ satisfies :

$$\frac{\partial^\alpha f}{(\partial t)^\alpha} = \frac{\partial^2 f}{(\partial x)^2} \qquad (x,t) \in \Pi \tag{11}$$

subject to boundary and initial values:

$$f(x,0) = 0, \quad \frac{\partial f(0,t)}{\partial x} = 0, \quad \frac{\partial f(1,t)}{\partial x} = \beta(u(t)), \quad \beta = \text{const}, \ \ f(x,T) = \nu. \tag{12}$$

We say that the control function $u(t)$ is admissible and denote $u(t) \in U_{ad}$, if the following conditions hold:

- $u(t)$ is a measurable function on $-1 \leq u(t) \leq 1$ for $t \in [0, T]$,
- $u(t)$ puts $f(x,T) = \nu$, in final time T where $x \in [0,1]$, ν is a suitable arbitrary constant with $|\nu| < 1$,
- $u(t)$ minimizes the following functional.

$$Min\ \ I(u(.)) = \int_0^T f_0(t, u(t))d(t), \tag{13}$$

where f_0 $C(\Omega)$ that is the space of all real-valued continuous functions on $\Omega = [0,T] \times [-1,1]$. If we set $f_0 = 1$, the above problem is denoted as minimum optimal time problem. Consider μ_k; $k = 1,2, \cdots$, be the sequence of positive roots of the equation $\mu_k tan \mu_k = \beta$.(see [2], [3] for more details)

By using relation (7) and multiply by $e^{\mu_k^2 t^{\frac{t^\alpha}{\alpha}}} cos(\mu_k x)$ and integrate over both side of (11) on the region Π and taking into account the boundary and initial conditions, one can see that the following equation:

$$\nu \int_0^1 cos(\mu_k x)dx = \beta cos(\mu_k) \int_0^T t^{\alpha-1} e^{-\mu_k^2 \frac{T^\alpha - t^\alpha}{\alpha}} u(t)dt \qquad k = 1,2, \cdots \tag{14}$$

Since $\mu_k tan(\mu_k) = \beta$ and replace with $\mu_k^2 = \lambda_k$, as a sequence of eigenvalues, the following relation is obtained:

$$\int_0^T t^{\alpha-1} e^{-\lambda_k \frac{T^\alpha - t^\alpha}{\alpha}} u(t)dt = \frac{\nu}{\lambda_k} \qquad k = 1,2, \cdots \tag{15}$$

Let $\phi_n(t,u) = t^{\alpha-1} e^{-\lambda_n \frac{T^\alpha - t^\alpha}{\alpha}} u(t)$ and $a_n = \frac{\nu}{\lambda_n}$ for $n = 1,2, \ldots$.

So, we have the following minimum time optimal control problem:

$$Min\ \ T = \int_0^T d(t), \tag{16}$$

subject to

$$\int_0^T \phi_n(t, u(t))d(t) = a(n). \qquad n = 1,2, \ldots \tag{17}$$

4 Metamorphosis

In general form, it is difficult to characterize the optimal control in U_{ad} ([11]), so we use a transformation to enlarge the set U_{ad}.

Let $u(.) \in U_{ad}$, the following mapping is considered:

$$\Lambda: \Theta \to \int_0^T \Theta(t, u(t))dt, \qquad \Theta \in C(\Omega), \tag{18}$$

where this mapping defines a positive linear functional on $C(\Omega)$, that identify each admissible $u(.)$ by $\Lambda(u)$. also the transformation $u \to \Lambda$ is an injection, see([9]). By (18), the problem (16)-(17) is rewritten following:

$$Min\ \ \Lambda(1), \tag{19}$$

subject to

$$\Lambda(\phi_n) = a(n). \qquad n = 1,2,\dots \tag{20}$$

By the Riesz representation theorem ([10]), every Radon measure Λ can be corresponding to regular, finite and unique Borel measure. So there exists a Borel measure μ on $M^+(\Omega)$; the space of all positive Radon measures on Ω; such that

$$\Lambda(\Theta) = \int_\Omega \Theta(t, u(t))dt = \mu(\Theta), \qquad \Theta \in C(\Omega), \tag{21}$$

These measures are required to have some properties, (see [11]) such as

$$|\mu(\Theta)| \leq T \sup_\Omega |\Theta(t, u(t))|, \tag{22}$$

So,

$$\mu(1) \leq T, \tag{23}$$

and by rewrite (20)

$$\mu(\phi_n) = a(n). \qquad n = 1,2,\dots. \tag{24}$$

Suppose that $\theta \in C(\Omega)$, where θ is not depend on u, i.e.

$$\theta(t, u_1) = \theta(t, u_2) \tag{25}$$

for all $t \in [0, T]$ and u_1, u_2 [-1, 1], where $u_1 u_2$, then the measure μ must satisfy

$$\mu(\theta) = \int_\Omega \theta d\mu = \int_0^T \theta(t, u)dt = \Delta\theta \tag{26}$$

where $\Delta\theta$ is the integral of $\theta(t, u(t))$ over $[0, T]$.

With the help of analysis used in ([11]) and topologize the space $M^+(\Omega)$, the set of all positive Radon measures on Ω, by the $weak^* - topology$, finally we have the following optimization problem in measure space:

$$Min \ \ \mu(1), \tag{27}$$

subject to

$$\mu \in Q \tag{28}$$

where Q as a compact subset of $M^+(\Omega)$, that is the intersection of all measures satisfied (23),(24) and (26) Simultaneously. see([11])

5 Approximation of the optimal measure

for solving the main problem (16)-(17), after the transformation the optimal time problem to minimize a linear form over a set of positive measures (27)-(28), here the optimal measure can be approximated by a finite-dimensional linear programming (LP) problem. The first step of process is considered the following problem in the space of all positive Radon measure:

$$Minimize \ \ I(\mu) = \int_\Omega d\mu \equiv \mu(1) \tag{29}$$

Subject to

$$\begin{cases} \mu(\phi_n) = a_n, & n = 1,2\cdots, \\ \mu(1) \leq T, \\ \mu(\theta) = \Delta\theta, & \theta \in C(\Omega) \end{cases} \tag{30}$$

Theorem 5.1. Consider $Q(M_1, M_2)$ be a subset of $M^+(\Omega)$ consisting of all measures which satisfy the problem of minimizing $\mu(1)$ over the set $Q(M_1, M_2)$ of measures in $M^+(\Omega)$

satisfying:

$$\begin{cases}\mu(\phi_n) = a_n, & n = 1,2\cdots, M_1 \\ \mu(1) \leq T, & \\ \mu(\theta_k) = \Delta\theta_k, & k = 1,2,\cdots, M_2\end{cases} \tag{31}$$

If $M_1 \to \infty$, $M_2 \to \infty$ then inf_Q(M_1,M_2)(1) inf_Q(1).

proof: see([11])

Now consider the following important theorem (see [11]).

Theorem 5.2. The optimal measure in the set $Q(M_1, M_2,)$ at which the functional $\mu \to \mu(1)$ attains its minimum has the form

$$\mu^* = \sum_{k=1}^{M_1+M_2} \alpha_k^* \delta(z_k^*) \tag{32}$$

where the coefficients $\alpha_k^* \geq 0$ and $z_k^* \in \Omega$ are unknowns for $k = 1,\cdots, M_1 + M_2$. Consider $\delta(z) \in M^+(\Omega)$ is a unitary atomic measure which is supported by the singleton set {z} and satisfied

$$\delta(z)f = f(z), \qquad f \in C(\Omega), \qquad z \in \Omega.$$

Thus by using (32), the problem (29)-(30) changes to the following nonlinear programming problem:

$$Min \quad \sum_{k=1}^{M_1+M_2} \alpha_k^* \tag{33}$$

subject to

$$\begin{cases} \sum_{k=1}^{M_1+M_2} \alpha_k^* \phi_n(z_k^*) = a_n \qquad n = 1,2,\ldots, M_1 \\ \sum_{k=1}^{M_1+M_2} \alpha_k^* \theta_k^j(z_k^*) = \Delta\theta_j \qquad j = 1,2,\ldots, M_2, \quad \sum_{k=1}^{M_1+M_2} \alpha_k^* \leq T, \\ \alpha_k^* \geq 0 \qquad k = 1,2,\ldots, M_1 + M_2. \end{cases} \tag{34}$$

where $z_k^* \in \Omega$.

Let $\omega = \{z_1, \cdots, z_N\}$ be a countable approximately dense subset of Ω. A measure $\mu^* \in M^+(\Omega)$ as a good approximation for μ can be found such that

$$\mu^* = \sum_{k=1}^{N} \alpha_k^* \delta(z_k), \tag{35}$$

where the coefficients α_k^* are the same as in the optimal measure μ^* in (32), and $z_k \in \omega, k = 1,2,\ldots, N$. (see [11]).

So we have the following problem:

$$Min \quad \sum_{k=1}^{N} \alpha_k \tag{36}$$

subject to

$$\begin{cases} \sum_{k=1}^{N} \alpha_k \phi_n(z_k) = a_n \qquad n = 1,2,\ldots, M_1 \\ \sum_{k=1}^{N} \alpha_k \theta_k^j(z_k) = \Delta\theta_j \qquad j = 1,2,\ldots, M_2, \quad \sum_{k=1}^{N+1} \alpha_k = T, \\ \alpha_k \geq 0 \qquad k = 1,2,\ldots, N+1. \end{cases} \tag{37}$$

where $z_i; i = 1,\ldots, N$ and $N >> M_1 + M_2$ for sufficiently large N in ω. Also it is mentioned that, we added α_{N+1} as a slack variable in corresponded constraint.

Now, we state the method of approximation T as optimal time by numerical method and (LP) problem:

It is to be noted that T is unknown in (36)-(37), so for approximation it, the time interval $[0,T]$ is divided into m_1 portions and $U=[-1,1]$ to m_2 portions, so $N=m_1m_2$, and consider $[0,T]$ into two parts: $[0,T]=[0,T_1]\cup[T_1,T]$.

Let T_1 is the best lower bound for T. By a search algorithm, which is proposed in ([12]), the best choice for this lower bound is selected. Also functions $\phi_n(t,u(t))$ appearing in (37) may be written as

$$\phi_n(t,u)=t^{\alpha-1}e^{-\lambda_n\frac{T^{\alpha}-t^{\alpha}}{\alpha}}u(t)=e^{\lambda_n\frac{t^{\alpha}}{\alpha}}e^{-\lambda_n\frac{T^{\alpha}}{\alpha}}u(t)t^{\alpha-1} \tag{38}$$

By Taylor series expansion in a neighborhood of $\frac{T^{\alpha}}{\alpha}$- $\frac{T_1^{\alpha}}{\alpha}$, one can see the following relation:

$$e^{-\lambda_n\frac{T^{\alpha}}{\alpha}}\approx e^{-\lambda_n\frac{T_1^{\alpha}}{\alpha}}(1-\lambda_n(\frac{T^{\alpha}}{\alpha}-\frac{T_1^{\alpha}}{\alpha})) \tag{39}$$

As a special case of choosing functionals on space Ω, the functions θ_k are introduced as a piecewise constant which are dependent only on the variable t:

$$\theta_k=\begin{cases}1 & t\in J_k=[\frac{(k-1)T_1}{m_1-1},\frac{kT_1}{m_1-1}] \qquad k=1,2,\cdots,m_1-1\\ 0 & o.w\end{cases} \tag{40}$$

and for $k=m_1$, we define $J_{m_1}=[T_1,T]$. so the right hand side of second equalities of (37), that is the integral of θ_k over $[0,T]$, is as follows:

$$\Delta\theta_k=\begin{cases}\frac{T_1}{m_1-1} & k=1,\cdots,m_1-1,\\ T-T_1 & k=m_1\end{cases} \tag{41}$$

By the (40) and (41), the second equalities of (37) is as follows:

$$\sum_{k=1}^{m_2}\alpha_k=\frac{T_1}{m_1-1},\sum_{k=m_2+1}^{2m_2}\alpha_k=\frac{T_1}{m_1-1},\cdots\sum_{(m_1-2)m_2+1}^{(m_1-1)m_2}\alpha_k=\frac{T_1}{m_1-1},\sum_{(m_1-1)m_2+1}^{(m_1m_2)}\alpha_k=T-T_1 \tag{42}$$

Adding the both side of equalities (42) leads to:

$$\sum_{k=1}^{N}\alpha_k=T \tag{43}$$

So the slack variable α_{k+1} in (37) equals zero.

Lastly the extended form of (LP) problem for finding optimal time is :

$$Min \quad \sum_{k=1}^{N}\alpha_k \tag{44}$$

subject to

$$\sum_{k=1}^{N}\alpha_k t_k^{\alpha-1}u_k e^{\lambda_n\frac{t_k^{\alpha}}{\alpha}}-\nu e^{\lambda_n\frac{T_1^{\alpha}}{\alpha}}\frac{T^{\alpha}}{\alpha}=-\nu e^{\lambda_n\frac{T_1^{\alpha}}{\alpha}}(\frac{T_1^{\alpha}}{\alpha}-\frac{1}{\lambda_n}) \qquad n=1,2,\dots,M_1\sum_{k=(i-1)m_2+1}^{im_2}\alpha_k=\frac{T_1}{m_1-1} \qquad i=1, \tag{45}$$

It is noteworthy that a piecewise-constant control function related to optimal time T is constructed such that the following numerical error:

$$E_1 = \| f(x,T) - \nu \|_2^2 \tag{46}$$

tends to zero.(see [2], [3], [11], [12])

6 Conclusion

In this paper the fractional heat equation in conformable definition sense is considered. The embedding process as a straightforward method is used to convert (FPDE) problem to (OCP) one's. Reformulating of heat equation in classical form to fractional form respect to time variable successfully is examined. (CFD) as a local fractional derivative is used to present a new and useful technique for solving (FPDE)'s. The authors believe that this method can be used in other (FPDE)'s especially in nonlinear fractional partial differential equations and in future works this subject will be investigated.

References

References

Gachpazan M and Kamyad A V 2004 Solving of second order nonlinear pde problems by using artificial controls with controlled error *J. Appl. Math. Computing* **15** 173

Gal'chukl I 1969 Optimal control of systems described by parabolic equations *J siam j. control* **7**

Effati S Nazemi A and Shabani H 2014 Time optimal control problem of the heat equation with thermal source *J IMA* **31** 385

Xingyang Y and Chuanju X 2013 Spectral optimization methods for the time fractional diffusion inverse problem *J. Numer. Math. Theor. Meth. Appl*. **6** 499

Mophou G. M 2011 Optimal control of fractional diffusion equation *J. Computer and mathematics with Applications.* **61** 68

Khalil R Horani M. Al Yousef A and Sababheh M 2014 A new definition of fractional derivative: *Journal of Computational and Applied Mathematics* **264** 65

Yavuz M 2017 Novel solution methods for initial boundary value problems of fractional order with conformable differentiation *J. IJOCTA* **8** 1

Ziaei E Farahi M.H and Safaie, E 2017 The approximate solution of nonlinear fractional optimal control problems by measure theory approach *J Progr. Fract. Differ. Appl.* doi:10.18576/pfda/paper

Farahi M. H Mehne H. H and Borzabadi A. H 2006 Wing drag minimization by using measure theory *J optim. methods soft* **21** 169

Royden H. L 1970 Real Analysis *London: The Macmillan Company*

Rubio J. E 1986 Control and Optimization: the Linear Treatment of Nonlinear problem: *Manchester University Press, Manchester and John Wiely, NewYork and Landon*

Mehne H. H Farahi M.H and Kamyad A.V 2006 MILP modelling for the time optimal control problem in the case of multiple targets *J optim. control appl. meth* **27** 77