



دومین سمینار ملی کنترل و بهینه‌سازی  
۱۳۹۷ آبان ۲۴-۲۵

The 2<sup>nd</sup> National Seminar on Control and Optimization  
15-16 November 2018



# An approximate method for solving fractional partial differential equation by using an embedding process

Ehsan Ziaei\* <sup>1</sup>, Mohammad Hadi Farahi <sup>2</sup>

<sup>1</sup>Salman institute of higher education, Mashhad, Iran.

<sup>2</sup>Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran and Salman institute of higher education, Mashhad, Iran

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## Abstract

In this article, solving of fractional partial differential equation FPDE with optimal time control approach is developed. Conformable derivative as a new definition of fractional derivative is considered. At first the FPDE is converted to optimal time control problem, then by using an embedding process, the obtained system is converted to finite-dimensional linear programming LP problem and finally optimal time corresponded by FPDE system is approximated. For example, this method is used on the conformable fractional heat equation with the initial and boundary conditions.

**Keywords:** Fractional partial differential equation, Optimal control problem, Embedding process, Conformable fractional derivative, Linear programming.

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## 1 Introduction

Partial differential equation PDE as an interest issue for modelling of natural phenomenal such as chemical physics, fluid flows and viscoelasticity is studied by many authors [1, 2, 3]. By extra information in fractional calculus a lot of abnormal physical systems such as fractional heat equation, fractional wave equation is described by FPDEs. In many works, Riemann-Liouville, Caputo and Grnwald-Letnikov fractional operators is applied to modelling and interpreting of system behaviors. Moreover, many methods for solving FPDE are based on the Laplace transform, Fourier series, Finite differences, Legendre wavelets and other numerical methods [4, 5]. Since most of the existing fractional operators are defined by singular kernels and non-local structure, they do not obey the basic chain and product rules. To overcome these drawbacks, in this work we present a novel approach to solve FPDEs by using conformable fractional derivative CFD, that is introduced by khalil [6]. CFD as a local and limit -based definition is expanded rapidly and used in many applications. However in this work all major initial and boundary conditions on FPDE is considered and embedding process, as a useful method is examined to solve this problem. As an advantage of the proposed method can be mentioned that, it is self-starting and not

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\*Speaker: eh.ziaei@mail.um.ac.ir

iterative, so this method has been developed to solve a variety of FPDE problems. Authors used CFD to solve Fractional optimal control problems successfully in previous works such as [8] and now are going to apply this definition for modelling a fractional heat equation in an unknown minimum time as an example of FPDEs. Using moment form and an embedding process respectively, problem can be approximated by finite-dimensional LP one's.

## 2 Preliminaries

In this section some basic concepts of conformable fractional derivative is mentioned. We assume  $w = f(t)$  ( $t \geq 0$ ) be a real valued and continuous function, and  $\alpha \geq 0$  is a given real number.

**Definition 2.1** Let  $f : [0, \infty] \rightarrow R$ , then the CFD of  $f(t)$  is defined as follows:(see[6])

$$T_\alpha f(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \quad 0 \leq \alpha < 1, t \geq 0. \quad (1)$$

We write sometimes  $f^{(\alpha)}(t)$  for  $T_\alpha f(t)$  to denote CFD of order  $\alpha$ , also if  $T_\alpha f(t)$  exists, then we say  $f$  is  $\alpha$ -differentiable.

Let  $\alpha \in (0, 1]$  and  $f, g$  be  $\alpha$ -differentiable for  $t > 0$ , then the following properties can be resulted by the CFD:

$$T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g), \quad a, b \in R, \quad (2)$$

$$T_\alpha(t^p) = pt^{p-\alpha}, \quad p \in R, \quad (3)$$

$$T_\alpha(\lambda) = 0, \quad \lambda = const, \quad (4)$$

$$T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f), \quad (5)$$

$$T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}. \quad (6)$$

Moreover, if  $f$  be a differentiable function, then one can prove that

$$T_\alpha f(t) = t^{1-\alpha} \frac{df(t)}{dt}. \quad (7)$$

**Definition 2.2** Let  $f : (0, t) \rightarrow R$  ( $t \geq 0$ ), be a continuous function and  $\alpha \in (0, 1)$ , the conformable  $\alpha$ -fractional integral of a function  $f$  is defined as:

$$I_\alpha f(t) = \int_0^t \tau^{\alpha-1} f(\tau) d\tau. \quad (8)$$

**Theorem 2.3** Let  $f : [0, \infty] \rightarrow R$  be a function such that  $f$  is differentiable and also  $\alpha$ -differentiable. Let  $g$  be a function defined in the range of  $f$  and also differentiable, then we have the following rule:

$$T_\alpha(fog)(t) = (T_\alpha f(g(t)) \cdot (T_\alpha g(t)) \cdot g(t)^{\alpha-1}. \quad (9)$$

**Theorem 2.4** Let  $f$  be a differentiable function for  $t > 0$ , and  $0 < \alpha \leq 1$ , then

$$I_\alpha T_\alpha f(t) = f(t) - f(0). \quad (10)$$

## 3 System description and formulation

Consider the following time fractional partial heat equation in conformable difinition sense: In the region  $\Pi = [0, 1] \times [0, T]$  and  $0 < \alpha \leq 1$ , the function  $f(x, t)$  satisfies :

$$\frac{\partial^\alpha f}{(\partial t)^\alpha} = \frac{\partial^2 f}{(\partial x)^2} \quad (x, t) \in \Pi \quad (11)$$

subject to boundary and initial values:

$$f(x, 0) = 0, \quad \frac{\partial f(0, t)}{\partial x} = 0, \quad \frac{\partial f(1, t)}{\partial x} = \beta(u(t)), \quad \beta = \text{const}, \quad f(x, T) = \nu. \quad (12)$$

We say that the control function  $u(t)$  is admissible and denote  $u(t) \in U_{ad}$ , if the following conditions hold:

- (i)  $u(t)$  is a measurable function on  $-1 \leq u(t) \leq 1$  for  $t \in [0, T]$ ,
- (ii)  $u(t)$  puts  $f(x, T) = \nu$ , in final time  $T$  where  $x \in [0, 1]$ ,  $\nu$  is a suitable arbitrary constant with  $|\nu| < 1$ ,
- (iii)  $u(t)$  minimizes the following functional.

$$\text{Min } I(u(\cdot)) = \int_0^T f_0(t, u(t)) dt, \quad (13)$$

where  $f_0 \in C(\Omega)$  that is the space of all real-valued continuous functions on  $\Omega = [0, T] \times [-1, 1]$ . If we set  $f_0 = 1$ , the above problem is denoted as minimum optimal time problem. Consider  $\mu_k$ ;  $k = 1, 2, \dots$ , are the sequence of positive roots of the equation  $\mu_k \tan \mu_k = \beta$ . (see [2, 3] for more details )

By using relation (7) and multiply by  $e^{\mu_k^2 \frac{t^\alpha}{\alpha}} \cos(\mu_k x)$  and integrate over both side of (11) on the region  $\Pi$  and taking into account the boundary and initial conditions, one can see that the following equation:

$$\nu \int_0^1 \cos(\mu_k x) dx = \beta \cos(\mu_k) \int_0^T t^{\alpha-1} e^{-\mu_k^2 \frac{T^\alpha - t^\alpha}{\alpha}} u(t) dt, \quad k = 1, 2, \dots \quad (14)$$

Since  $\mu_k \tan(\mu_k) = \beta$  and replace with  $\mu_k^2 = \lambda_k$ , as a sequence of eigenvalues [2], the following relation is obtained:

$$\int_0^T t^{\alpha-1} e^{-\lambda_k \frac{T^\alpha - t^\alpha}{\alpha}} u(t) dt = \frac{\nu}{\lambda_k} \quad k = 1, 2, \dots \quad (15)$$

Let  $\phi_n(t, u) = t^{\alpha-1} e^{-\lambda_n \frac{T^\alpha - t^\alpha}{\alpha}} u(t)$  and  $a_n = \frac{\nu}{\lambda_n}$  for  $n = 1, 2, \dots$

So, we have the following minimum time optimal control problem:

$$\text{Min } T = \int_0^T dt \quad (16)$$

subject to

$$\int_0^T \phi_n(t, u(t)) dt = a(n). \quad n = 1, 2, \dots \quad (17)$$

## 4 Metamorphosis

In general form, it is difficult to characterize the optimal control in  $U_{ad}$  (see [9]), so we use a transformation to enlarge the set  $U_{ad}$ .

Let  $u(\cdot) \in U_{ad}$ , the following mapping is considered:

$$\Lambda : \Theta \longrightarrow \int_0^T \Theta(t, u(t)) dt, \quad \Theta \in C(\Omega), \quad (18)$$

where this mapping defines a positive linear functional on  $C(\Omega)$ , that identify each admissible  $u(\cdot)$  by  $\Lambda(u)$ . also the transformation  $u \longrightarrow \Lambda$  is an injection, [9]. By (18), the problem (16)-(17) is rewritten as follows:

$$\text{Min } \Lambda(1), \quad (19)$$

subject to

$$\Lambda(\phi_n) = a(n), \quad n = 1, 2, \dots \quad (20)$$

By the Riesz representation theorem (see[10]), every Radon measure  $\Lambda$  can be corresponding to regular, finite and unique Borel measure. So there exists a Borel measure  $\mu$  on  $M^+(\Omega)$ ; the space of all positive Radon measures on  $\Omega$ ; such that

$$\Lambda(\Theta) = \int_{\Omega} \Theta(t, u(t)) dt = \mu(\Theta), \quad \Theta \in C(\Omega), \quad (21)$$

These measures are required to have some properties, [11] such as

$$|\mu(\Theta)| \leq T \sup_{\Omega} |\Theta(t, u(t))|, \quad (22)$$

So,

$$\mu(1) \leq T, \quad (23)$$

and by rewriting (20)

$$\mu(\phi_n) = a(n), \quad n = 1, 2, \dots \quad (24)$$

Suppose that  $\theta \in C(\Omega)$ , where  $\theta$  is not depend on  $u$ , i.e.

$$\theta(t, u_1) = \theta(t, u_2), \quad (25)$$

for all  $t \in [0, T]$  and  $u_1, u_2 \in [-1, 1]$ , where  $u_1 \neq u_2$ , then the measure  $\mu$  must satisfy

$$\mu(\theta) = \int_{\Omega} \theta d\mu = \int_0^T \theta(t, u) dt = \Delta\theta \quad (26)$$

where  $\Delta\theta$  is the integral of  $\theta(t, u(t))$  over  $[0, T]$ .

With the help of analysis used in [11] and topologize the space  $M^+(\Omega)$ , the set of all positive Radon measures on  $\Omega$ , by the *weak\* - topology*, finally we have the following optimization problem in measure space:

$$\text{Min } \mu(1), \quad (27)$$

subject to

$$\mu \in Q \quad (28)$$

where  $Q$  as a compact subset of  $M^+(\Omega)$ , that is the intersection of all measures satisfied (23),(24) and (26) Simultaneously. (see[11])

## 5 Approximation of the optimal measure

for solving the main problem (16)-(17), after the transformation the optimal time problem to minimize a linear form over a set of positive measures (27)-(28), here the optimal measure can be approximated by a finite-dimensional linear programming LP problem. The first step of process is consider the following problem in the space of all positive Radon measure:

$$\text{Minimize } I(\mu) = \int_{\Omega} d\mu \equiv \mu(1) \quad (29)$$

Subject to

$$\begin{cases} \mu(\phi_n) = a_n, & n = 1, 2, \dots, \\ \mu(1) \leq T, \\ \mu(\theta) = \Delta\theta, & \theta \in C(\Omega). \end{cases} \quad (30)$$

**Theorem 5.1.** Consider  $Q(M_1, M_2)$  be a subset of  $M^+(\Omega)$  consisting of all measures which satisfy the problem of minimizing  $\mu(1)$  over the set  $Q(M_1, M_2)$  of measures in  $M^+(\Omega)$  satisfying:

$$\begin{cases} \mu(\phi_n) = a_n, & n = 1, 2, \dots, M_1 \\ \mu(1) \leq T, \\ \mu(\theta_k) = \Delta\theta_k, & k = 1, 2, \dots, M_2 \end{cases} \quad (31)$$

If  $M_1 \rightarrow \infty, M_2 \rightarrow \infty$  then  $\inf_{Q(M_1, M_2)} \mu(1) \rightarrow \inf_Q \mu(1)$ .

proof: see[11]

Now consider the following important theorem ( see [11]).

**Theorem 5.2.** The optimal measure in the set  $Q(M_1, M_2,)$  at which the functional  $\mu \rightarrow \mu(1)$  attains its minimum has the form

$$\mu^* = \sum_{k=1}^{M_1+M_2} \alpha_k^* \delta(z_k^*) \quad (32)$$

where the coefficients  $\alpha_k^* \geq 0$  and  $z_k^* \in \Omega$  are unknowns for  $k = 1, \dots, M_1 + M_2$ . Consider  $\delta(z) \in M^+(\Omega)$  is a unitary atomic measure which is supported by the singleton set  $\{z\}$  and satisfied

$$\delta(z)f = f(z), \quad f \in C(\Omega), \quad z \in \Omega.$$

Thus by using (32), the problem (29)-(30) changes to the following nonlinear programming problem:

$$\text{Min} \quad \sum_{k=1}^{M_1+M_2} \alpha_k^* \quad (33)$$

subject to

$$\begin{cases} \sum_{k=1}^{M_1+M_2} \alpha_k^* \phi_n(z_k^*) = a_n & n = 1, 2, \dots, M_1 \\ \sum_{k=1}^{M_1+M_2} \alpha_k^* \theta_k^j(z_k^*) = \Delta\theta_j & j = 1, 2, \dots, M_2, \\ \sum_{k=1}^{M_1+M_2} \alpha_k^* \leq T, \\ \alpha_k^* \geq 0 & k = 1, 2, \dots, M_1 + M_2. \end{cases} \quad (34)$$

where  $z_k^* \in \Omega$ .

Let  $\omega = \{z_1, \dots, z_N\}$  be a countable approximately dense subset of  $\Omega$ . A measure  $\mu^* \in M^+(\Omega)$  as a good approximation for  $\mu$  can be found such that

$$\mu^* = \sum_{k=1}^N \alpha_k^* \delta(z_k), \quad (35)$$

where the coefficients  $\alpha_k^*$  are the same as in the optimal measure  $\mu^*$  in (32), and  $z_k \in \omega, k = 1, 2, \dots, N$ . (see [11]).

So we have the following problem:

$$\text{Min} \quad \sum_{k=1}^N \alpha_k \quad (36)$$

subject to

$$\begin{cases} \sum_{k=1}^N \alpha_k \phi_n(z_k) = a_n & n = 1, 2, \dots, M_1 \\ \sum_{k=1}^N \alpha_k \theta_k^j(z_k) = \Delta\theta_j & j = 1, 2, \dots, M_2, \\ \sum_{k=1}^{N+1} \alpha_k = T, \\ \alpha_k \geq 0 & k = 1, 2, \dots, N + 1. \end{cases} \quad (37)$$

where  $z_i; i = 1, \dots, N$  and  $N \gg M_1 + M_2$  for sufficiently large  $N$  in  $\omega$ . Also it is mentioned that, we added  $\alpha_{N+1}$  as a slack variable in corresponded constraint.

Now, we state the method of approximation  $T$  as optimal time by numerical method and LP problem:

It is to be noted that  $T$  is unknown in (36)-(37), so for approximation it, the time interval  $[0, T]$  is divided into  $m_1$  portions and  $U = [-1, 1]$  to  $m_2$  portions, so  $N = m_1 m_2$ , and consider  $[0, T]$  into two parts:  $[0, T] = [0, T_1] \cup [T_1, T]$ .

Let  $T_1$  is the best lower bound for  $T$ . By a search algorithm, which is proposed in [12], the best choice for this lower bound is selected.

Also functions  $\phi_n(t, u(t))$  appearing in (37) may be written as

$$\phi_n(t, u) = t^{\alpha-1} e^{-\lambda_n \frac{T^\alpha - t^\alpha}{\alpha}} u(t) = e^{\lambda_n \frac{t^\alpha}{\alpha}} e^{-\lambda_n \frac{T^\alpha}{\alpha}} u(t) t^{\alpha-1} \quad (38)$$

By Taylor series expansion in a neighborhood of  $\frac{T^\alpha}{\alpha} - \frac{T_1^\alpha}{\alpha}$  for  $0 < \alpha \leq 1$ , one can see the following relation:

$$e^{-\lambda_n \frac{T^\alpha}{\alpha}} \approx e^{-\lambda_n \frac{T_1^\alpha}{\alpha}} \left(1 - \lambda_n \left(\frac{T^\alpha}{\alpha} - \frac{T_1^\alpha}{\alpha}\right)\right). \quad (39)$$

As a special case of choosing functionals on space  $\Omega$ , the functions  $\theta_k$  are introduced as a piecewise constant which are dependent only on the variable  $t$ :

$$\theta_k = \begin{cases} 1 & t \in J_k = \left[\frac{(k-1)T_1}{m_1-1}, \frac{kT_1}{m_1-1}\right] \quad k = 1, 2, \dots, m_1 - 1, \\ 0 & o.w. \end{cases} \quad (40)$$

and for  $k = m_1$ , we define  $J_{m_1} = [T_1, T]$ . So the right hand side of second equalities of (37), that is the integral of  $\theta_k$  over  $[0, T]$ , is as follows:

$$\Delta\theta_k = \begin{cases} \frac{T_1}{m_1-1} & k = 1, \dots, m_1 - 1, \\ T - T_1. & k = m_1. \end{cases} \quad (41)$$

By the (40) and (41), the second equalities of (37) is as follows:

$$\sum_{k=1}^{m_2} \alpha_k = \frac{T_1}{m_1-1}, \quad \sum_{k=m_2+1}^{2m_2} \alpha_k = \frac{T_1}{m_1-1}, \dots, \sum_{(m_1-2)m_2+1}^{(m_1-1)m_2} \alpha_k = \frac{T_1}{m_1-1}, \quad \sum_{(m_1-1)m_2+1}^{(m_1 m_2)} \alpha_k = T - T_1, \quad (42)$$

Adding the both side of equalities (42) leads to:

$$\sum_{k=1}^N \alpha_k = T. \quad (43)$$

So the slack variable  $\alpha_{N+1}$  in (37) equals zero.

Lastly the extended form of LP problem for finding optimal time is :

$$Min \quad \sum_{k=1}^N \alpha_k \quad (44)$$

subject to

$$\begin{aligned} \sum_{k=1}^N \alpha_k t_k^{\alpha-1} u_k e^{\lambda_n \frac{t_k^\alpha}{\alpha}} - \nu e^{\lambda_n \frac{T_1^\alpha}{\alpha}} \frac{T^\alpha}{\alpha} &= -\nu e^{\lambda_n \frac{T_1^\alpha}{\alpha}} \left(\frac{T^\alpha}{\alpha} - \frac{1}{\lambda_n}\right) \quad n = 1, 2, \dots, M_1, \\ \sum_{k=(i-1)m_2+1}^{im_2} \alpha_k &= \frac{T_1}{m_1-1} \quad i = 1, 2, \dots, m_1 - 1, \\ \sum_{k=(m_1-1)m_2+1}^N \alpha_k - T &= -T_1, \\ \alpha_k \geq 0, \quad T > 0, \quad 0 < \alpha \leq 1 \quad &k = 1, 2, \dots, N. \end{aligned} \quad (45)$$

According to the boundary condition in (12), the piecewise-constant control function is obtained from the solving of (44)-(45) is constructed such that the following numerical error:

$$E_1 = \| f(x, T) - \nu \|_2^2 \quad (46)$$

tends to zero.(see [3, 12]). The method of approximation optimal control is outlined in [3].

## 6 Conclusion

In this paper the fractional heat equation in conformable definition sense is considered. The embedding process as a straightforward method is used to convert FPDE problem to OCP one's. Reformulating of heat equation in classical form to fractional form respect to time variable successfully is examined. CFD as a local fractional derivative is used to present a new and useful technique for solving FPDEs. The authors believe that this method can be used in other FPDEs especially in nonlinear fractional partial differential equations and in future works this subject will be investigated.

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