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On Fractional Infinite-Horizon Optimal Control Problems

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Abstract

In this paper a method based on neural networks to solve fractional infinite-horizon optimal control problems (FIHOCP)s is presented where the dynamic control system depends on Caputo fractional derivatives. First, with the help of an approximation, the Caputo derivative is replaced to integer order derivative. Using a suitable change of variable, the IHOCP is transformed to a finite-horizon one. According to the Pontryagin minimum principle (PMP) for optimal control problems and by constructing an error function, an unconstrained minimization problem is defined. In the optimization problem, trial solutions for state, costate and control functions are used where these trial solutions are constructed by using two-layered perceptron neural network. Two numerical results are given to show the efficiency.

Keywords: Fractional infinite-horizon problems, Caputo fractional derivative, Pontryagin minimum principle, Optimal control problem, Neural networks, Optimization.

1 Introduction

In this paper a novel optimal control problem in Fractional calculus as FIHOCPs is presented. The important novelty of this paper is that a fractional dynamic system is appeared in an optimal control problem, where the problem is defined on $[0, \infty)$. Based on some mentioned advantages of neural networks for computing, we provide a hybrid method based on neural network scheme for solving FIHOCP. It should be noted that in solving the FIHOCP with neural networks, we can replace the fractional derivative with integer order derivative with a good approximation. The Riemann-Liouville derivative is expandable in a power series involving integer order derivatives only. To approximate this derivative, a second approach was carried out in [1], where a good approximation is obtained without the requirement of such higher-order smoothness on the admissible functions.

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2 Preliminaries

In [1] a good approximation is obtained without the requirement of such higher-order smoothness on the admissible functions. The method can be explained, for left Riemann-Liouville fractional derivatives, in the following way

$$\begin{aligned} {}_a^R D_t^\alpha (y(t)) &= A(\alpha)(t-a)^{-\alpha} y(t) + B(\alpha)(t-a)^{1-\alpha} \dot{y}(t) \\ &- \textstyle\sum_{p=2}^{\infty} C(\alpha,p)(t-a)^{1-p-\alpha} y_p(t), \end{aligned} \tag{1}$$

where $\alpha \in (0,1)$ and $y \in C^2[a,b]$. In (1), $y_p(t)$ is the solution of the system

$$\begin{cases} \dot{y}_p(t) = (1-p)(t-a)^{p-2} y(t), \\ y_p(a) = 0, \end{cases} \tag{2}$$

for $p = 2, 3, \ldots$. A similar formula can be deduced for the Caputo fractional derivative by using relationship between Riemann-Liouville and Caputo fractional derivatives as:

$${}_a^c D_t^\alpha (y(t)) = {}_a^R D_t^\alpha (y(t)) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha}. \tag{3}$$

For computational purposes, we truncate the sum and consider the finite expansion

$$\begin{aligned} {}_a^R D_t^\alpha (y(t)) &\approx A(\alpha,k)(t-a)^{-\alpha} y(t) + B(\alpha,k)(t-a)^{1-\alpha} \dot{y}(t) \\ &- \textstyle\sum_{p=2}^{k} C(\alpha,p)(t-a)^{1-p-\alpha} y_p(t), \end{aligned} \tag{4}$$

where $k \geq 2$ and

$$A(\alpha,k) = \frac{1}{\Gamma(1-\alpha)} \left[1 + \sum_{p=2}^{k} \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha)(p-1)!} \right], \qquad B(\alpha,k) = \frac{1}{\Gamma(2-\alpha)} \left[1 + \sum_{p=1}^{k} \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!} \right].$$

$$C(\alpha,p) = \frac{\Gamma(p-1+\alpha)}{\Gamma(2-\alpha)\Gamma(\alpha-1)(p-1)!}.$$

3 Fractional infinite-horizon optimal control problem

A FIHOCP can be defined as follows

$$\text{Minimize } J = \int_0^\infty f(t, x(t), u(t))\, \mathrm{d}t, \tag{5}$$

$$\text{subject to} \begin{cases} {}_0^c D_t^\alpha (x(t)) = g(t, x(t), u(t)), & t \in [0, \infty), \\ x(0) = x_0, \quad \lim_{t\to\infty} x(t) = x_f, \end{cases} \tag{6}$$

where

$$x \ : [0,\infty) \to \mathbb{R}^n, \quad u \ : [0,\infty) \to \mathbb{R}^l.$$

We first replace the operator ${}_0^c D_t^\alpha (x(t))$ with the help of approximation (4). For simplicity of our discussion, we assume $A = A(\alpha,k)$, $B = B(\alpha,k)$ and $C_p = C(\alpha,p)$. Thus

$$\dot{x}(t) = \frac{g(t,x(t),u(t)) - At^{-\alpha} x(t) + \sum_{p=2}^{k} C_p t^{1-p-\alpha} y_p(t) + \frac{x(0)t^{-\alpha}}{\Gamma(1-\alpha)}}{Bt^{1-\alpha}}.$$

We also define

$$Y(t) = (y_2(t), y_3(t), \ldots, y_k(t)),$$

and

$$G(t,x(t),Y(t),u(t))=\frac{g(t,x(t),u(t))-At^{-\alpha}x(t)+\sum_{p=2}^{k}C_pt^{1-p-\alpha}y_p(t)+\frac{x(0)t^{-\alpha}}{\Gamma(1-\alpha)}}{Bt^{1-\alpha}}.$$

Thus, the OCP (5) and (6) is reduced as

$$\text{Minimize } J=\int_0^\infty f(t,x(t),u(t))\,\mathrm{d}t, \tag{7}$$

$$\text{subject to}\begin{cases}\dot{x}(t)=G(t,x(t),Y(t),u(t)),\\ \dot{y}_p(t)=(1-p)t^{p-2}x(t), & p=2,\dots,k,\\ x(0)=x_0, \quad \lim_{t\to\infty}x(t)=x_f,\\ y_p(0)=0, & p=2,\dots,k.\end{cases} \tag{8}$$

4 Transformation of the infinite-horizon into a finite one

The following time transformation is introduced in order into transform the infinite horizon problem to a finite-horizon one defined on $\tau\in[0,1)$ as

$$t=\frac{\tau}{1-\tau}. \tag{9}$$

Using the proposed transformation, the infinite horizon problem (5) and (6) is replaced by

$$\text{Minimize } J=\int_{[0,1)}\frac{1}{(1-\tau)^2}f(\frac{\tau}{1-\tau},X(\tau),U(\tau))\,\mathrm{d}\tau, \tag{10}$$

$$\text{subject to}\begin{cases}\dot{X}(\tau)=\frac{1}{(1-\tau)^2}G(\frac{\tau}{1-\tau},X(\tau),\mathcal{Y}(\tau),U(\tau)),\\ \dot{\mathcal{Y}}_p(\tau)=\frac{1}{(1-\tau)^2}(1-p)(\frac{\tau}{1-\tau})^{p-2}X(\tau), & p=2,\dots,k,\\ X(0)=x_0, \quad \lim_{\tau\to 1^-}X(\tau)=x_f,\\ \mathcal{Y}_p(0)=0, & p=2,\dots,k,\end{cases} \tag{11}$$

where

$$X(\tau)=x(\frac{\tau}{1-\tau}),\quad U(\tau)=u(\frac{\tau}{1-\tau}),\quad \mathcal{Y}(\tau)=Y(\frac{\tau}{1-\tau}). \tag{12}$$

To find the optimal control, a Hamiltonian function for the problem (10) and (11) is as

$$H=\frac{1}{(1-\tau)^2}\Big[f(\frac{\tau}{1-\tau},X,U)+\lambda G(\frac{\tau}{1-\tau},X,\mathcal{Y},U)+\sum_{p=2}^{k}\gamma_p((1-p)(\frac{\tau}{1-\tau})^{p-2}X)\Big], \tag{13}$$

where λ and γ_p, $p=2,\dots,k$ are the Lagrange multipliers. For simplicity we define $\Upsilon(\tau)=(\gamma_2(\tau),\gamma_3(\tau),\dots,\gamma_k(\tau))$, $\mathcal{Y}=(\mathcal{Y}_2,\mathcal{Y}_3,\dots,\mathcal{Y}_k)$ and $H=H(X,\mathcal{Y},U,\lambda,\Upsilon,\tau)$. We use trial solutions for the state, control and Lagrange multiplier functions as

$$\begin{cases}n_x=\sum_{i=1}^{I}\nu_i^x\sigma(z_i^x),\quad z_i^x=w_i^x\tau+b_i^x, & n_y=\sum_{i=1}^{I}\nu_i^y\sigma(z_i^y),\quad z_i^y=w_i^y\tau+b_i^y,\\ n_u=\sum_{i=1}^{I}\nu_i^u\sigma(z_i^u),\quad z_i^u=w_i^u\tau+b_i^u, & n_\lambda=\sum_{i=1}^{I}\nu_i^\lambda\sigma(z_i^\lambda),\quad z_i^\lambda=w_i^\lambda\tau+b_i^\lambda,\\ n_\Upsilon=\sum_{i=1}^{I}\nu_i^\Upsilon\sigma(z_i^\Upsilon),\quad z_i^\Upsilon=w_i^\Upsilon\tau+b_i^\Upsilon,\end{cases} \tag{14}$$

where σ is a sigmoid function [2]. The trial solutions are selected as

$$\bar{X}=x_0+\tau(1-\tau)n_x+\tau(x_f-x_0),\quad \bar{\mathcal{Y}}=\tau n_y,\quad \bar{U}=n_u,\quad \bar{\lambda}=n_\lambda,\quad \bar{\Upsilon}=(1-\tau)n_\Upsilon. \tag{15}$$

The trial solutions (15) of feed forward neural network are the universal approximation and must satisfy in necessary optimality conditions of the problem (10) and (11) as

$$\frac{\sigma\bar{H}}{\sigma\bar{\lambda}}=\dot{\bar{X}}(\tau),\quad \frac{\sigma\bar{H}}{\sigma\bar{\Upsilon}}=\dot{\bar{\mathcal{Y}}}(\tau),\quad \frac{\sigma\bar{H}}{\sigma\bar{X}}=-\dot{\bar{\lambda}}(\tau),\quad \frac{\sigma\bar{H}}{\sigma\bar{\mathcal{Y}}}=-\dot{\bar{\Upsilon}}(\tau),\quad \frac{\sigma\bar{H}}{\sigma\bar{U}}=0 \tag{16}$$

where $\bar{H} = H(\tau, \bar{X}, \bar{\mathcal{Y}}, \bar{U}, \bar{\lambda}, \bar{\Upsilon})$. An error function corresponds to the system (16) may be constructed as

$$E(\Omega) = \frac{1}{2} \sum_{j=1}^{m} \left\{ E_1(\tau_j, \Omega) + E_2(\tau_j, \Omega) + E_3(\tau_j, \Omega) + E_4(\tau_j, \Omega) + E_5(\tau_j, \Omega) \right\}, \tag{17}$$

where $\Omega = (w_x, w_y, w_u, w_\lambda, w_\Upsilon, b_x, b_y, b_u, b_\lambda, b_\Upsilon, \nu_x, \nu_y, \nu_u, \nu_\lambda, \nu_\Upsilon)$ and

$$\begin{cases} E_1(\tau, \Omega) = \left[\frac{\sigma \bar{H}}{\sigma \bar{\lambda}} - \dot{\bar{X}}(\tau)\right]^2, \qquad E_2(\tau, \Omega) = \left[\frac{\sigma \bar{H}}{\sigma \bar{\Upsilon}} - \dot{\bar{\mathcal{Y}}}(\tau)\right]^2, \\ E_1(\tau, \Omega) = \left[\frac{\sigma \bar{H}}{\sigma \bar{X}} + \dot{\bar{\lambda}}(\tau)\right]^2, \qquad E_2(\tau, \Omega) = \left[\frac{\sigma \bar{H}}{\sigma \bar{\mathcal{Y}}} + \dot{\bar{\Upsilon}}(\tau)\right]^2, \qquad E_5(\tau, \Omega) = \left[\frac{\sigma \bar{H}}{\sigma \bar{U}}\right]^2. \end{cases} \tag{18}$$

In this way, the first problem is transformed into an unconstrained optimization problem which can be solved by any classical mathematical optimization algorithms.

5 Numerical examples

This section is devoted to illustrate the efficiency of the proposed numerical approach. Consider the problem

$$\text{Minimize } \frac{1}{2} \int_0^\infty (x_1^2(t) + 4u^2(t))\, dt,$$

$$\text{subject to} \begin{cases} {}_0^c D_t^\alpha (x_1(t)) = x_2(t), \\ {}_0^c D_t^\alpha (x_2(t)) = -x_2(t) + u(t), \\ x_1(0) = x_2(0) = 0.1, \lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_2(t) = 0. \end{cases}$$

The optimal trajectory of the problem for $\alpha = 1$ is given in [3] as

$$x_1(t) = [0.1 + (0.1 + \tfrac{0.1}{\sqrt{2}})t] \exp\left(\tfrac{-t}{\sqrt{2}}\right), \qquad x_2(t) = [0.1 - (0.1 + \tfrac{0.1}{\sqrt{2}})t] \exp\left(\tfrac{-t}{\sqrt{2}}\right).$$

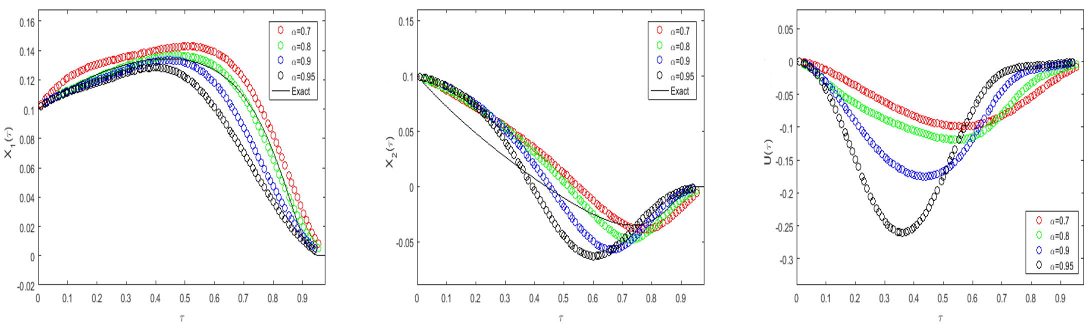

Figure 1: Optimal states $X_1(.)$, $X_2(.)$ and control $U(.)$ for different values of α

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