



Characterization of continuous K - g -frames in Hilbert C^* -modules

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ABSTRACT. This note is focused on merging the notions of g -frame and K -frame in a Hilbert C^* -module U for an operator K that belongs to $End_{\mathcal{A}}^*(U)$. Here, $\mathcal{A}$ refers to a complex C^* -algebra and $End_{\mathcal{A}}^*(U)$ refers to the collection of all adjointable $\mathcal{A}$ -linear maps on U . This introduces the concept of a continuous $K - g$ -frame for Hilbert C^* -modules, and provides some characterization.

Keywords: Continuous $K - g$ -frame, Hilbert C^* -module.

AMS Mathematical Subject Classification [2010]: 42C15; 46L99.

1. Introduction

Frank and Larson [4] extended the concept of a frame for an operator on a Hilbert C^* -module. Since then, various generalizations of frames in Hilbert C^* -modules have gained more attention. For instance, you can refer to [1, 5, 7] and the bibliographies therein to learn more about these generalization. The concept of a g -frame for a Hilbert C^* -module introduced in [4, 5] and the concept of a K -frame in a Hilbert C^* -module, for an operator $K \in End_{\mathcal{A}}^*(\mathcal{H})$ defined in [6], where $\mathcal{A}$ is a complex C^* -algebra with the norm $\|\cdot\|_{\mathcal{A}}$ and $End_{\mathcal{A}}^*(\mathcal{H})$ denotes the set of all adjointable $\mathcal{A}$ -linear maps on a Hilbert $\mathcal{A}$ -module $\mathcal{H}$. In what follows, the purpose is to define the concept of a c - K - g -frame for Hilbert C^* -modules and extend some results to the situation of c - K - g -frames in Hilbert C^* -modules, of course, there have been some discussions in this regard previously [2]. In our study, we will use I as an indexing set and $\mathbb{N}$ and $\mathbb{C}$ as the sets of natural numbers and complex numbers, respectively. We will also use U and V as two Hilbert C^* -modules and $\{V_k\}_{k \in I}$ as a family of subspaces of V . For every $k \in I$, $End_{\mathcal{A}}^*(U, V_k)$ denotes the class of all adjointable $\mathcal{A}$ -linear maps from U into V_k . Additionally, $End_{\mathcal{A}}^*(U)$ is abbreviated for $End_{\mathcal{A}}^*(U, U)$. The notation $R(K)$ for the range of operator K is used. Finally, we assume that $(\mathcal{M}, \mu)$ is a measure space with a positive measure μ .

Recall that a pre-Hilbert $\mathcal{A}$ -module $\mathcal{X}$ is a (left) $\mathcal{A}$ -module equipped with an $\mathcal{A}$ -valued inner product. The norm on $\mathcal{X}$ is defined by $\|f\|_{\mathcal{X}} = \|\langle ff \rangle\|_{\mathcal{A}}^{1/2}$ for all $f \in \mathcal{X}$. If $\mathcal{X}$ is a Banach space with the norm $\|\cdot\|_{\mathcal{X}}$, then $\mathcal{X}$ is said to be a Hilbert $\mathcal{A}$ -module or Hilbert C^* -module over $\mathcal{A}$. For a given C^* -algebra $\mathcal{A}$, let $\bigoplus_{m \in \mathcal{M}} V_m$ be the Hilbert $\mathcal{A}$ -module is defined by

$$\bigoplus_{m \in \mathcal{M}} V_m := \left\{ \{g_m\}_{m \in \mathcal{M}} : g_m \in V_m, \left\| \int_{\mathcal{M}} |g_m|^2 d\mu(m) \right\| < \infty \right\},$$

where for all $f = \{f_m\}_{m \in \mathcal{M}}$, $g = \{g_m\}_{m \in \mathcal{M}} \in \bigoplus_{m \in \mathcal{M}} V_m$, $\mathcal{A}$ -valued inner product is given by

$$\langle f, g \rangle = \int_{\mathcal{M}} \langle f_m, g_m \rangle d\mu(m).$$

DEFINITION 1.1. Let U be a Hilbert C^* -module and $(\mathcal{M}, \mu)$ be a measure space with positive measure μ . A continuous frame, or simply a c -frame for U with respect to $(\mathcal{M}, \mu)$ is a family $\{f_m\}_{m \in \mathcal{M}}$ of vectors in U for which

- (i) For all $f \in U$, the mapping $m \mapsto \langle f, f_m \rangle$ is measurable on $\mathcal{M}$.

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(ii) There exist constants $A, B > 0$ such that

$$A\langle f, f \rangle \leq \int_{\mathcal{M}} \langle f, f_m \rangle \langle f_m, f \rangle d\mu(m) \leq B\langle f, f \rangle, \forall f \in U.$$

The constants A, B are called c -frame bounds. The family $\{f_m\}_{m \in \mathcal{M}}$ is called a c -Bessel system with Bessel bound B if at least the right-hand of the above inequality is satisfied.

DEFINITION 1.2. A family $\{\Lambda_m\}_{m \in \mathcal{M}} \subset \text{End}_{\mathcal{A}}^*(U, V_m)$ is said to be a continuous g -frame, or simply a c - g -frame for U with respect to $\{V_m\}_{m \in \mathcal{M}}$ if the following hold:

- (i) For all $f \in U$, the mapping $m \mapsto \langle f, \Lambda_m f \rangle$ is measurable on $\mathcal{M}$.
- (ii) There exist constants $A, B > 0$ such that

$$A\langle f, f \rangle \leq \int_{\mathcal{M}} \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) \leq B\langle f, f \rangle, \forall f \in U.$$

The constants A, B are called c - g -frame bounds. If $A = B$, the family $\{\Lambda_m\}_{m \in \mathcal{M}}$ is said to be a tight c - g -frame for U . A Parseval c - g -frame for U with respect to $\{V_m\}_{m \in \mathcal{M}}$ is a tight c - g -frame whenever $A=1$.

2. c - K - g -frames in Hilbert C^* -modules

In this section, we introduce the concept of a c - K - g -frame for a Hilbert C^* -module and derive results on c - K - g -frames.

DEFINITION 2.1. Suppose $(\mathcal{M}, \mu)$ is a measure space with positive measure μ and $K \in \text{End}_{\mathcal{A}}^*(U)$. We say that $\{\Lambda_m\}_{m \in \mathcal{M}} \subset \text{End}_{\mathcal{A}}^*(U, V_m)$ a continuous K - g -frame, or simply a c - K - g -frame for U with respect to $\{V_m\}_{m \in \mathcal{M}}$ if the following conditions hold:

- (i) For all $f \in U$, the mapping $m \mapsto \langle f, \Lambda_m f \rangle$ is measurable on $\mathcal{M}$.
- (ii) There exist constants $A, B > 0$ such that

$$(1) \quad A\langle K^* f, K^* f \rangle \leq \int_{\mathcal{M}} \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) \leq B\langle f, f \rangle, \forall f \in U.$$

The constants A, B are called c - K - g -frame bounds. We say that $\{\Lambda_m\}_{m \in \mathcal{M}}$ is a tight c - K - g -frame if there exists a constant $A > 0$ such that

$$\int_{\mathcal{M}} \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) = A\langle K^* f, K^* f \rangle, \forall f \in U.$$

A Parseval c - K - g -frame for U with respect to $\{V_m\}_{m \in \mathcal{M}}$ is a tight c - K - g -frame whenever $A = 1$. We call the family $\{\Lambda_m\}_{m \in \mathcal{M}}$ a c - g -Bessel system for U with respect to $\{V_m\}_{m \in \mathcal{M}}$ with c - g -Bessel bound B if at least the right-hand inequality of (1) is satisfied.

REMARK 2.2. If $K = I_U$ (identity operator on U), then every c - K - g -frame for U is a c - g -frame. Also, if $\mathcal{M} = \mathbb{N}$ and μ is the counting measure on $\mathcal{M}$, then any c - K - g -frame is a K - g -frame for U .

REMARK 2.3. Every c - g -frame for U with respect to $\{V_m\}_{m \in \mathcal{M}}$ is a c - K - g -frame for U . In fact, if $\{\Lambda_m\}_{m \in \mathcal{M}}$ is a c - g -frame for U , then by the definition there exist constants $A, B > 0$ such that

$$A\langle f, f \rangle \leq \int_{\mathcal{M}} \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) \leq B\langle f, f \rangle, \forall f \in U.$$

On the other hand, using $\langle K^* f, K^* f \rangle \leq \|K\|^2 \langle f, f \rangle$, $\forall f \in U$, we obtain

$$A\|K\|^{-2} \langle K^* f, K^* f \rangle \leq \int_{\mathcal{M}} \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) \leq B\langle f, f \rangle, \forall f \in U.$$

Therefore, $\{\Lambda_m\}_{m \in \mathcal{M}}$ is a c - K - g -frame for U with respect to $\{V_m\}_{m \in \mathcal{M}}$.

DEFINITION 2.4. Suppose that $K \in \text{End}_{\mathcal{A}}^*(U)$ and $\{\Lambda_m\}_{m \in \mathcal{M}} \subset \text{End}_{\mathcal{A}}^*(U, V_m)$ is a c - K - g -frame for U with respect to $\{V_m\}_{m \in \mathcal{M}}$. We define the linear operator S as follows

$$S : U \longrightarrow U, \quad Sf = \int_{\mathcal{M}} \Lambda_m^* \Lambda_m f d\mu(m), \quad \forall f \in U.$$

The operator S is called the c - g -frame operator of $\{\Lambda_m\}_{m \in \mathcal{M}}$.

It is known that, S is a positive, bounded and self-adjoint operator. Furthermore, we can conclude that the operator S satisfies:

$$\langle Sf, g \rangle = \int_{\mathcal{M}} \langle \Lambda_m^* \Lambda_m f, g \rangle d\mu(m), \quad \forall f, g \in U.$$

REMARK 2.5. Note that, generally, similar to K -frames, the c - g -frame operator S of a c - K - g -frame for a Hilbert C^* -module U is not invertible on U . However, if K has closed range, then the c - g -frame operator S on $R(K)$ is a positive, self-adjoint, and invertible operator.

Assume that $\{\Lambda_m\}_{m \in \mathcal{M}} \subset \text{End}_{\mathcal{A}}^*(U, V_m)$ is a c - g -frame for U with respect to $\{V_m\}_{m \in \mathcal{M}}$ with the c - g -frame operator S . Similar to the usual frames, the family $\{\Lambda_m S^{-\frac{1}{2}}\}_{m \in \mathcal{M}}$ is a Parseval c - g -frame for U with respect to $\{V_m\}_{m \in \mathcal{M}}$. For a given $K \in \text{End}_{\mathcal{A}}^*(U)$, we denote $T_K(U)$ as the class of all tight c - K - g -frames for a Hilbert C^* -module U .

THEOREM 2.6. Suppose that $K_1, K_2 \in \text{End}_{\mathcal{A}}^*(U)$ and U is a finitely or countably generated Hilbert C^* -module. Then $T_{K_1}(U) \subseteq T_{K_2}(U)$ if and only if $K_1 K_1^* = \lambda K_2 K_2^*$ for some $\lambda > 0$.

PROPOSITION 2.7. Every adjointable operator K on a unital Hilbert C^* -module U admits a c - K - g -frame for U .

THEOREM 2.8. Assume $K_1, K_2 \in \text{End}_{\mathcal{A}}^*(U)$ with $R(K_1^*) \perp R(K_2^*)$. If $\{\Lambda_m\}_{m \in \mathcal{M}} \subset \text{End}_{\mathcal{A}}^*(U, V_m)$ is a c - K_1 - g -frame as well as a c - K_2 - g -frame for U with respect to $\{V_m\}_{m \in \mathcal{M}}$ and α, β are scalars, then $\{\Lambda_m\}_{m \in \mathcal{M}}$ is a c - $(\alpha K_1 + \beta K_2)$ - g -frame and a c - $K_1 K_2$ - g -frame for U with respect to $\{V_m\}_{m \in \mathcal{M}}$.

The next results provide a characterization of c - K - g -frames.

THEOREM 2.9. Let $K \in \text{End}_{\mathcal{A}}^*(U)$ and $\{\Lambda_m\}_{m \in \mathcal{M}} \subset \text{End}_{\mathcal{A}}^*(U, V_m)$ such that for any $f \in U$, the mapping $m \mapsto \langle f, \Lambda_m f \rangle$ is measurable on $\mathcal{M}$. Suppose the operator $T^* : U \mapsto \bigoplus_{m \in \mathcal{M}} V_m$ defined by $T^* f = \{\Lambda_m f\}_{m \in \mathcal{M}}$ and $\overline{\text{Ran}}(T^*)$ is orthogonally complemented. Then, $\{\Lambda_m\}_{m \in \mathcal{M}}$ is a c - K - g -frame for U with respect to $\{V_m\}_{m \in \mathcal{M}}$ if and only if there are constants $A, B > 0$ such that

$$(2) \quad A \|K^* f\|^2 \leq \left\| \int_{\mathcal{M}} \langle \Lambda_m f, \Lambda_m f \rangle d\mu(m) \right\| \leq B \|f\|^2, \quad \forall f \in U.$$

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