یازدهمین سمینار آنالیز هارمونیک و کاربردها
۱۲ و ۱۳ اردیبهشت ماه ۱۴۰۳
دانشگاه کردستان
11th Seminar on Harmonic Analysis and Applications
1 - 2 May 2024
University of Kurdistan

Inductive Limits in Matrix Quasi Ordered Vector Spaces

G. H. Esslamzadeh
Department of Mathematics, Faculty of Sciences, Shiraz University, Shiraz, Iran
M. Asgari Chafochayi and F. Taleghani *
Department of Mathematics, Lahijan Branch, Islamic Azad University, Lahijan, Iran

Abstract. Inductive limits, if they exist in a category, play an important role to yield more new objects. In this paper we show the category of matrix quasi ordered vector spaces is a category with inductive limits. Then we construct the inductive limit in the category of Archimedean quasi ordered vector spaces, independent of Archimedeanization process, too.
Keywords: Quasi ordered $*$-vector spaces, Matrix quasi ordered $*$-vector spaces, Inductive limit, State space.
AMS Mathematical Subject Classification [2010]: 46L07, 46B40.

1. Introduction

The concept of inductive limit that has been found useful in applications such as quantum physics (for example see [6]), is one of the fundamental constructions of C^*-algebras. On the other hand, a systematic investigation of inductive limits of operator systems appeared in [7] by constructing the inductive limits in the category of matrix ordered vector spaces.

The first and third authors in [4] through a process of characterizing C^*-algebras and operator systems, showed that properness of positive cone is not necessary in some of the results in the theory of operator algebras. Moreover, Dosive studied unital self-adjoint linear subspaces of some special classes of unbounded operators on Hilbert spaces as concrete local operator systems [2] and proved its equivalence with the definition of local operator systems, that is, $*$-vector spaces with a family of cones (not necessarily proper) and with an Archimedean matrix order unit.

In this paper we continue our study of inductive limit's structure in categories with not necessarily proper cone, which we began in [1]. This helps us to understand more about the structure of quantum systems as abstract versions of local operator systems where the cones involved are genuinely nonproper[3].

Throughout, *$*$-vector space* means a complex vector space $\mathcal{X}$ with an additive involution whose set of self-adjoint elements is denoted with $\mathcal{X}_{sa}$. A *cone* in $\mathcal{X}$ is a nonempty subset $C \subseteq \mathcal{X}$ which is closed under addition and multiplication by nonnegative reals. If moreover C satisfies the identity $C \bigcap (-C) = \{0\}$, then it is called a *proper cone*. We equip $\mathcal{X}$ with a fixed cone $\mathcal{X}^+ \subseteq \mathcal{X}_{sa}$ which we call the *positive cone* of $\mathcal{X}$. This cone naturally induces a reflexive and transitive but not necessarily antisymmetric relation $\leq$ on $\mathcal{X}_{sa}$ by $x \leq y$ if $y - x \in \mathcal{X}^+$, $(x, y \in \mathcal{X}_{sa})$ which we call the *quasi order* induced by $\mathcal{X}^+$ and we call $\mathcal{X}$ a *quasi ordered $*$-vector space*.

An element $e \in \mathcal{X}^+$ is called an *order unit* for $\mathcal{X}$ if for each $x \in \mathcal{X}_{sa}$, there exists a positive real number r such that $x \leq re$. The order unit e is called *Archimedean* and the triple $(\mathcal{X}, \mathcal{X}^+, e)$ is called *Archimedean quasi ordered $*$-vector space*, if whenever $x \in \mathcal{X}_{sa}$ and $x + re \geq 0$ for every positive real number r, then $x \geq 0$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be $*$-vector spaces. The notation $M_n(\mathcal{X})$ denotes the set of all $n \times n$ matrices with entries in $\mathcal{X}$. The natural addition, scalar multiplication and canonical $*$-operation $[a^{kl}]^* = [(a^{lk})^*]$ turn $M_n(\mathcal{X})$ into a $*$-vector space. Let $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ be the space of linear maps from $\mathcal{X}$ into $\mathcal{Y}$ and $\phi \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. For each $n \in \mathbb{N}$ the n-th amplification of ϕ is the map $\phi^{(n)} : M_n(\mathcal{X}) \longrightarrow M_n(\mathcal{Y})$ given by $\phi^{(n)}([x^{ij}]) = [\phi(x^{ij})]$.

*speaker

We say that $(\mathcal{X}, \{C_n\}_{n=1}^{\infty})$ is a *matrix quasi ordered $*$-vector space* if $\mathcal{X}$ is a $*$-vector space in which the following conditions hold:

(i) C_n is a cone in $M_n(\mathcal{X})$ for each $n \in \mathbb{N}$;

(ii) For each $m, n \in \mathbb{N}$ and each $\Lambda \in M_{n,m}(\mathbb{C})$ we have $\Lambda^* C_n \Lambda \subseteq C_m$ (Compatibility condition).

Moreover if $\mathcal{X}$ contains an element e such that $e \otimes I_n$ is an order unit for $M_n(\mathcal{X})$, $(n \in \mathbb{N})$, then e is called a matrix order unit.

Let $(\mathcal{X}, C_n, e)$ and $(\mathcal{Y}, D_n, e')$ be matrix quasi ordered $*$-vector spaces with order units e and e', respectively. We say that $\phi : \mathcal{X} \longrightarrow \mathcal{Y}$ is *positive* [resp. *self adjoint* or *involutive / unital*] if $\phi(x) \in \mathcal{Y}^+ = D_1$ for all $x \in \mathcal{X}^+ = C_1$ [resp. $\phi(x^*) = \phi(x)^*$ for all $x \in \mathcal{X}$ / $\phi(e) = e'$]. If $\phi^{(n)}$ is positive for all $n \in \mathbb{N}$, we say that ϕ is *completely positive*. We denote the space of all completely positive linear maps from $\mathcal{X}$ into $\mathcal{Y}$ by $CP(\mathcal{X}, \mathcal{Y})$. A linear functional $f : \mathcal{X} \longrightarrow \mathbb{C}$ is called a *state* if it is unital and positive. The set $S(\mathcal{X})$ of all states on $\mathcal{X}$, which is called the *state space* of $\mathcal{X}$, is not empty if and only if e is nonnegative [5]. Set $\mathcal{X}_{Arch} = \mathcal{X}/N$ where

$$N = \bigcap \{ker f : f \in S(\mathcal{X})\},$$

and

$$\mathcal{X}_{Arch}^+ = \{a + N : re + a \in \mathcal{X}^+ \text{ for all } r > 0\}.$$

The pair $(\mathcal{X}_{Arch}, \mathcal{X}_{Arch}^+)$ is an ordered $*$-vector space with Archimedean order unit $e + N$ which is called the *Archimedeanization* of $\mathcal{X}$ [5].

Let I be a directed set, $\{\mathcal{X}_i : i \in I\}$ be a family of objects of a category $\mathcal{C}$ and $\{f_{i,j} : \mathcal{X}_i \longrightarrow \mathcal{X}_j : i, j \in I, i \leq j\}$ be a family of morphisms in $\mathcal{C}$ with the properties $f_{i,i} = id_{\mathcal{X}_i}$ and $f_{i,k} = f_{j,k} \circ f_{i,j}$ for all $i, j, k \in I$ where $i \leq j \leq k$. Then the pair $(\{\mathcal{X}_i\}, \{f_{i,j}\}_{i \leq j})_{i,j \in I}$ is called a *direct system*. The *inductive limit* of this system, which is denoted by $\varinjlim \mathcal{X}_i$, is an object $\mathcal{X}$ in $\mathcal{C}$ together with morphisms $\phi_i : \mathcal{X}_i \longrightarrow \mathcal{X}$, $(i \in I)$ satisfying the properties

(i) $\phi_i = \phi_j \circ f_{i,j}$ for all $i, j \in I$ that $i \leq j$ and

(ii) the pair $(\mathcal{X}, \{\phi_i\}_{i \in I})$ is universal in the sense that for any other such pair $(\mathcal{Y}, \{\psi_i\}_{i \in I})$, there exists a unique morphism $\psi : \mathcal{X} \longrightarrow \mathcal{Y}$ making the diagram commute for all $i \in I$. A category in which every direct system has an inductive limit is called a category with inductive limits.

Let a pair $(\{\mathcal{X}_i\}, \{f_{i,j}\}_{i \leq j})_{i,j \in I}$ be an arbitrary direct system in the category of quasi ordered vector spaces [1]. That is, for every $i \in I$, $(\mathcal{X}_i, \mathcal{X}_i^+)$ is a quasi ordered $*$-vector space with nonnegative order unit e_i and $f_{i,j}$'s are unital positive maps. Then define

$$\mathcal{X}^0 = \{x \in \prod_{i \in I} \mathcal{X}_i \ : \ \exists h \in I \text{ such that } f_{i,j}(x(i)) = x(j) \text{ for all } j \geq i \geq h\}$$

and

$$N^0 = \{x \in \mathcal{X}^0 : \exists h \in I \text{ such that } x(i) = 0 \text{ for all } i \geq h\}.$$

Set $\mathcal{X} = \mathcal{X}^0/N^0$ and define linear maps $\phi_h : \mathcal{X}_h \longrightarrow \mathcal{X}$ for all $h \in I$ by $\phi_h(a) = x + N^0$ where

$$x : I \longrightarrow \bigcup_{i \in I} \mathcal{X}_i, \quad x(i) = f_{h,i}(a) \text{ if } i \geq h, \text{ and } x(i) = 0 \text{ otherwise.}$$

So $\mathcal{X}$ has a quasi order structure induced by

$$\mathcal{X}^+ := \{\phi_i(a) : a \in \mathcal{X}_i, \text{ and } f_{i,j}(a) \in \mathcal{X}_j^+ \text{ for some } j \geq i\}. \tag{1}$$

The pair $(\mathcal{X}, \mathcal{X}^+)$ with the net of unital positive maps $\{\phi_i\}$ is an inductive limit of $(\{\mathcal{X}_i\}, \{f_{i,j}\}_{i \leq j})_{i,j \in I}$ [1, Theorem 1].

2. Main results

Convension. The category $\mathcal{MQ}$ consists of matrix quasi ordered $*$-vector spaces with nonnegative matrix order unit as objects and unital completely positive linear maps as morphisms. Suppose that a family of objects $\{(\mathcal{X}_i, \{C_n^i\}, e_i)\}_{i \in I}$ with morphisms $\{f_{i,j} : \mathcal{X}_i \longrightarrow \mathcal{X}_j\}_{i \leq j}$ is a given inductive system in the category of matrix quasi ordered $*$-vector spaces.

For any $i \in I$, since $(\mathcal{X}_i, \{C_n^i\}, e_i)$, is a matrix quasi ordered $*$-vector space, then for all $n \in \mathbb{N}$,

$(M_n(\mathcal{X}_i), C_n^i)$ is a quasi ordered $*$-vector space with nonnegative order unit $e_i^{(n)} = e_i \otimes I_n$ such that $\{C_n^i\}_{n\in\mathbb{N}}$ is a compatible sequence of positive cones and $f_{i,j}^{(n)} : M_n(\mathcal{X}_i) \longrightarrow M_n(\mathcal{X}_j)$ is a unital positive linear map for all $j \geq i$.
Let $n \in \mathbb{N}$ be arbitrary. It is clear that $f_{i,i}^{(n)} = id_{M_n(\mathcal{X}_i)}$ and $f_{j,k}^{(n)} o f_{i,j}^{(n)} = f_{i,k}^{(n)}$ for $i \leq j \leq k$, $i, j, k \in I$. Then $(\{M_n(\mathcal{X}_i)\}_{i\in I}, \{f_{i,j}^{(n)}\}_{i\leq j})$ is a direct system in the category of quasi ordered $*$-vector spaces with the direct limit $\varinjlim_i M_n(\mathcal{X}_i)$ and the related family of unital positive maps $\{\phi_i^n : M_n(\mathcal{X}_i) \longrightarrow \varinjlim_i M_n(\mathcal{X}_i)\}_{i\in I}$ so that by (1) the positive cone is equal to

$$(\varinjlim_i M_n(\mathcal{X}_i))^+ = \bigcup_{i\in I} \{\phi_i^n(A) : A \in M_n(\mathcal{X}_i),\ \exists j \geq i, f_{i,j}^{(n)}(A) \in (C_n^j)^+\} \tag{2}$$

and from unitality of morphisms relative to inductive limits in the category of quasi ordered vector spaces [1], the order unit is equal to $\phi_i^n(e_i \otimes I_n)$ for all $i \in I$.

LEMMA 2.1. *Using notation $\mathcal{X}$ for $\varinjlim M_1(\mathcal{X}_i)$ and $\phi_i : \mathcal{X}_i \longrightarrow \mathcal{X}$ for ϕ_i^1, we have unital positive maps $\phi_i^{(n)} : M_n(\mathcal{X}_i) \longrightarrow M_n(\mathcal{X})$ with*

$$\phi_i^{(n)}([a^{kl}]_{k,l=1}^n) = [\phi_i(a^{kl})]_{k,l=1}^n \qquad (n \in \mathbb{N})$$

and so $M_n(\mathcal{X}) = \bigcup_{i\in I} \phi_i^{(n)}(M_n(\mathcal{X}_i))$.

This lemma helps us to prove the involutive correspondence between codomains of maps $\phi_i^{(n)}$ and ϕ_i^n.

PROPOSITION 2.2. *For all $n \in \mathbb{N}$, the $*$-vector spaces $M_n(\mathcal{X})$ and $\varinjlim M_n(\mathcal{X}_i)$ are isomorphic via the mapping $\pi_n : M_n(\mathcal{X}) \longrightarrow \varinjlim M_n(\mathcal{X}_i)$ given by $\pi_n o \phi_i^{(n)} = \phi_i^n$.*

Now we're ready to define the desired matrix quasi order construction on the quasi ordered $*$-vector space $(\mathcal{X}, \mathcal{X}^+, e)$.

DEFINITION 2.3. For each $n \in \mathbb{N}$, set

$$C_n := \pi_n^{-1}((\varinjlim M_n(\mathcal{X}_i))^+).$$

Since π_n's are $*$-vector space isomorphisms, then by (2),

$$C_n = \bigcup_{i\in I} \{\phi_i^{(n)}(A) : A \in M_n(\mathcal{X}_i),\ \exists j \geq i, f_{i,j}^{(n)}(A) \in (C_n^j)^+\}$$

and so $\{C_n\}_{n\in\mathbb{N}}$ is a compatible family of positive cones on $\{M_n(\mathcal{X})\}_{n\in\mathbb{N}}$. Moreover for each $n \in \mathbb{N}$, $e \otimes I_n$ is a matrix order unit on $(M_n(\mathcal{X}), C_n)$ so that the new object in the category of matrix quasi ordered $*$-vector space is concluded: The triple $(\mathcal{X}, \{C_n\}_{n\in\mathbb{N}}, e)$ with the family of completely positive maps $\{\phi_i\}_{i\in I}$ is inductive limit of the given inductive system $(\{\mathcal{X}_i\}_{i\in I}, \{f_{i,j}\}_{i\leq j})$ in the category of matrix quasi ordered vector spaces.

THEOREM 2.4. *The category $\mathcal{MQ}$ is a category with inductive limits.*

REMARK 2.5. Suppose $\mathcal{AQ}$ is the category whose objects are quasi ordered $*$-vector spaces with nonnegative Archimedean order units and whose morphisms are unital positive linear maps. Let $(\{\mathcal{X}_i\}, \{f_{i,j}\}_{i\leq j})_{i,j\in I}$ be a direct system in this category and let $\mathcal{F}$ be the forgetful functor from $\mathcal{AQ}$ to the category of quasi ordered vector spaces which was introduced in the preceding section. Notice that the inductive limit of $\{\mathcal{F}(\mathcal{X}_i)\}$ in the category of quasi ordered vector spaces, that was identified in the previous section and was denoted by $\mathcal{X}$, is not necessarily Archimedean (See [7] for a counterexample). So in [1, section 3] we discussed its Archimedeanization and we show that it is equal to $\varinjlim \mathcal{X}_i$ in $\mathcal{AQ}$.

But here we define an independent structure for $\varinjlim \mathcal{X}_i$ in $\mathcal{AQ}$. Set

$$M_{\mathbb{R}} = \{x \in (\mathcal{X}^0)_{sa} : \forall r > 0,\ \exists i \in I,\ \forall j \geq i\ \ re_j + x(j) \in \mathcal{X}_j^+ \text{ and } re_j - x(j) \in \mathcal{X}_j^+\}$$

and $M = M_{\mathbb{R}} \oplus iM_{\mathbb{R}}$. Consider $\mathcal{X}^0/M$ with the positive cone

$$\{x + M : x \in (\mathcal{X}^0)_{sa},\ \forall r > 0,\ \exists i \in I,\ \forall j \geq i\ \ re_j + x(j) \in \mathcal{X}_j^+\}.$$

THEOREM 2.6. *By preceding remark assumptions, we can define a net of unital positive maps from $\mathcal{X}_i$'s to construct inductive limit of $(\{\mathcal{X}_i\}, \{f_{i,j}\}_{i \leq j})_{i,j \in I}$.*

References

1. M. Asgari Chafochayi, G. H. Esslamzadeh and F. Taleghani, *Inductive Limits in Quasi Ordered Vector Spaces*, To appear in the Iranian Journal of Science
2. A. A. Dosiev, *Local operator spaces, unbounded operators and multinormed C*-algebras*, J. Func. Anal. **255** (2008), 1724-1760
3. A. A. Dosi, *Quantum system structures of quantum spaces and entanglement breaking maps*, Sb. Math. **210** (2019), 21-93
4. G. H. Esslamzadeh and F. Taleghani, *Structure of quasi operator systems*, Linear Algebra Appl. **438** (2013), 1372-1392
5. G. H. Esslamzadeh, M. Moazami Goodarzi and F. Taleghani, *Structure of quasi ordered $*$-vector spaces* , Iran. J. Sci. Technol. Trans. A **38**(4) (2014), 445-453
6. M. Fannes, B. Nachtergaele and R. F. Werner, *Finitely correlated states on quantum spin chains* , Commun. Math. Phys. **144** (1992), 443-490
7. L. Mawhinney and I.G. Todorov, *Inductive limits in the operator system and related categories*, Diss. Math. **536** (2018), 1-57

E-mail: esslamz@shirazu.ac.ir

E-mail: marziye.asgari1360ch@gmail.com

E-mail: fatemeh.taleghani@gmail.com