پانزدهمین سمینار احتمال
و فرآیندهای تصادفی
۸ و ۹ شهریور ۱۴۰۴
دانشگاه کردستان

15th Seminar On Probability and Stochastic Processes
August 30-31, 2025
University of Kurdistan

انجمن آمار ایران

Goodness of Fit Test Based on Extropy and Dirichlet Process

Shamim Haghshenas Jazi*, Zahra Saberi†

[1]Department of Mathematical Sciences, Isfahan University of Technology, Isfahan, Iran

Abstract

In this paper, we propose a novel goodness of fit test based on extropy from a Bayesian nonparametric perspective. Extropy, introduced as a complement to entropy, focuses on the structured concentration within a probability distribution rather than on uncertainty. To enhance estimation accuracy, especially with small samples, we employ a Bayesian estimator built upon the Dirichlet process prior. We define a relative extropy as a measure of distance between the empirical distribution and a hypothesized model, and construct a testing framework based on the Relative Belief Ratio (RB) and Strength indices to quantify evidence for or against the null hypothesis. Some simulation studies have been done in order to investigate the proposed approach.

Keywords: Bayesian Nonparametric Methods, Dirichlet Process, Extropy, Relative Belief Ratio, Strength.

Mathematics Subject Classification (2020): 62F15, 62G10, 94A17.

1 Introduction

Assessing the fit between observed data and theoretical models is a cornerstone of statistical analysis. Traditional measures such as entropy have long served to quantify uncertainty. Grzegorzewski and Wieczorkowski (1999) introduced an entropy-based goodness-of-fit test, which has inspired further exploration of uncertainty measures in model assessment.

In 2015, Lad and Sanfilippo and Agro (2015) introduced extropy, a complementary concept to entropy that emphasizes the structured concentration within probability distributions. Following this, Qiu and Jia (2018) proposed two nonparametric estimators of extropy based on order statistics.

Meanwhile, classical goodness-of-fit tests like Kolmogorov-Smirnov and Anderson-Darling remain widely used but suffer from limited sensitivity to various types of deviations. This has motivated the search for more flexible, robust alternatives.

*M.S in Statistics, shamim.haghshenas78@gmail.com
†Assistant Professor of Statistics, z_saberi@iut.ac.ir

To provide a Bayesian nonparametric framework for goodness-of-fit testing, one can combine extropy estimation with the Dirichlet process prior Ferguson (1973). This approach uses relative extropy as a distance metric and incorporates the Relative Belief Ratio and its calibration measure, Strength, as discussed in the relative belief framework Evans (2015); Al-Labadi and Evans (2018).

The remainder of the paper is organized as follows:
In Section 2, a bayesian nonparametric estimator of extropy is obtained. Section 3 discusses using the developed estimator for goodness of fit tests.

2 Extropy and Bayesian Nonparametric Estimation

Extropy is a measures of uncertainty that serves as a complementry of entropy Lad and Sanfilippo and Agro (2015). For a random variable with density function $f(x)$, it is defined as:

$$J(f) = -\frac{1}{2}\int f^2(x)\,dx. \tag{1}$$

Unlike entropy, which focuses on uncertainty, extropy highlights how a distribution organizes its probability mass. Several estimators Qiu and Jia (2018) have been proposed for estimating can be rewritten as

$$J(f) = -\frac{1}{2}\int_0^1 \left[\frac{d}{dt}F^{-1}(t)\right]^{-1} dt. \tag{2}$$

Now, if $\mathbf{X} = (X_1, \ldots, X_n)$ is a random sample from F and $X_{(i)}$'s, $i = 1, \ldots, n$, are the order statistics according to $\boldsymbol{X}$, then (2) can be approximated as follow:

$$JQ = -\frac{1}{2n}\sum_{i=1}^{n} \frac{c_i \cdot m/n}{X_{(i+m)} - X_{(i-m)}}, \tag{3}$$

where m is the window size (typically proportional to $\sqrt{n}$)), and c_i is the boundary correction coefficient defined as:

$$c_i = \begin{cases} 1 + \frac{i-1}{m}, & \text{if } 1 \le i \le m, \\ 2, & \text{if } m+1 \le i \le n-m, \\ 1 + \frac{n-i}{m}, & \text{if } n-m+1 \le i \le n. \end{cases}$$

2.1 Nonparametric Bayesian Estimation for Extropy

For Bayesian nonparametric estimation of extropy, we use the Dirichlet prior. Let $\mathrm{DP}(\alpha, G)$ be the Dirichlet process with a positive real number α and a fixed probability measure G Ferguson (1973). Now, let $F \sim \mathrm{DP}(\alpha, G)$. By the conjugacy property of the Dirichlet process, the posterior distribution of F given an observed sample $\mathbf{x} = (x_1, \ldots, x_n)$, generated from F, is $\mathrm{DP}(\alpha+n, G_{\alpha,n})$, where

$$G_{\alpha,n} = \frac{\alpha}{\alpha+n}G + \frac{n}{\alpha+n}F_n$$

and F_n is the empirical cumulative distribution function (ECDF) of the observed sample x. By using the properties of the Dirichlet process, if $P \sim \mathrm{DP}(\alpha, G)$, then we have the following representation:

$$P = \frac{\sum_{i=1}^{\infty} L^{-1}(\Gamma_i)\,\delta_{Y_i}}{\sum_{i=1}^{\infty} L^{-1}(\Gamma_i)},$$

where $\Gamma_i = E_1 + \cdots + E_i$; $E_i \overset{i.i.d.}{\sim} \mathrm{Exp}(1)$, $Y_i \overset{i.i.d.}{\sim} G$, independent of Γ_i. Define the function:

$$L(x) = a\int_x^\infty \frac{1}{t}e^{-t}\,dt, \quad x > 0,$$

with its inverse given by:

$$L^{-1}(y) = \inf\{x > 0 \mid L(x) \le y\}.$$

Now, by this property, we can estimate $\left(\frac{d}{dt}F^{-1}(t)\right)^{-1}$ by the inverse of the slope of the straight line connecting the following points:

$$\left(P_N(Y_{(i-m)}) = \sum_{k=1}^{i-m} P_{k,N}, Y_{(i-m)}\right) \quad \text{and} \quad \left(P_N(Y_{(i+m)}) = \sum_{k=1}^{i+m} P_{k,N}, Y_{(i+m)}\right).$$

This equal to:

$$\frac{C_{i,a}}{Y_{(i+m)} - Y_{(i-m)}},$$

where

$$c_{i,a} = \begin{cases} \sum_{k=2}^{i+m} p_{k,N} & 1 \le i \le m, \\ \sum_{k=i-m+1}^{i+m} p_{k,N} & m+1 \le i \le N-m, \\ \sum_{k=i-m+1}^{N} p_{k,N} & N-m+1 \le i \le N. \end{cases}$$

Note that $E(C_{i,a}) = \frac{m}{N}c_i$, where c_i is defined in (3).

3 Goodness of Fit Testing

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a random sample from distribution F. The main goal is to test the hypothesis $H_0 : F = F_0$, where F_0 is a specified distribution. Now let $F \sim \mathrm{DP}(\alpha, G)$ be the prior on F, for some choice of α and base measure G. Then the posterior distribution is:

$$F \mid x \sim \mathrm{DP}(\alpha + n, G_x)$$

If F_0 is true, then the posterior distribution of the distance d between F and F_0 must be more concentrated around 0 than the prior distribution of this distance.

As a distance for this purpose, we consider relative extropy distance Ferguson (1973), which is defined as follows:

$$d(f||g) = \frac{1}{2}\int (f-g)^2\,dx.$$

A Bayesian nonparametric estimation for d is given by:

$$\hat{d} = \hat{d}(f||g) = -J_{m,N,a}(f) - J(g) - \sum_{i=1}^{N} p_{i,N}g(Y_i)$$

where $J_{m,N,\alpha}(f)$ is the Bayesian estimator of the extropy of f, and $J(g) = -\frac{1}{2}\int g^2(x)\,dx$ is the extropy of the hypothesized model g.

We can show that if $N \to \infty$, $m \to \infty$, $\frac{m}{N} \to 0$, and $a \to \infty$, then $\hat{d} \xrightarrow{P} d(f||g)$.

3.1 Relative Belief Ratio

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a sample of $f(x|\theta)$, where $\theta \sim \pi(\theta)$. For a hypothesis testing $H_0 : \theta = \theta_0$, the relative belief ratio is defined as:

$$RB(\theta_0 \mid \mathbf{x}) = \frac{\pi(\theta_0 \mid \mathbf{x})}{\pi(\theta_0)}.$$

This is measuring how our belief changed that θ_0 is the true value from a prior to a posterior and If $RB > 1$, there is evidence in favor of θ_0; if $RB < 1$, there is evidence against θ_0 and if $RB = 1$, there is no evidence either way.

For calibrating this measure, Evans (2015) suggested to computing the posterior probability as bellow and named it *strength*;

$$\text{Strength} = \Pi(RB(\theta|x) \leq RB(\theta_0|x) \mid x),$$

Interpretation of the Strength

We distinguish two main cases:

Case 1: $RB_\theta(\theta_0 \mid x) > 1$ (data support H_0)

- If $Str_\theta(\theta_0)$ is large, it means that there is a small chance that the true value of θ corresponds to a higher relative belief ratio. Thus, there is strong evidence in favor of H_0.
- If $Str_\theta(\theta_0)$ is small, it indicates a high probability that the true θ value has a larger relative belief ratio than θ_0. Consequently, $RB_\theta(\theta_0 \mid x)$ only provides weak evidence in favor of H_0.

Case 2: $RB_\theta(\theta_0 \mid x) < 1$ (data are against H_0)

- If $Str_\theta(\theta_0)$ is small, there is strong evidence against H_0, as the true value of θ likely has a higher relative belief ratio than θ_0.
- If $Str_\theta(\theta_0)$ is large, only weak evidence against H_0 is provided by $RB_\theta(\theta_0 \mid x)$.

Classification of the Strength of $RB_\theta(\theta_0 \mid x)$

Based on the value of $Str_\theta(\theta_0)$, we categorize the strength into three levels:

- If $Str_\theta(\theta_0) \leq 0.25$, the evidence is considered weak.
- If $0.25 < Str_\theta(\theta_0) \leq 0.75$, the evidence is considered moderate.
- If $0.75 < Str_\theta(\theta_0) \leq 1$, the evidence is considered strong.

It is important to note that although $Str_\theta(\theta_0)$ shares some similarities with a p-value, it is not intended for making direct acceptance or rejection decisions for H_0. Instead, it measures the strength of the relative belief ratio for evaluating H_0.

4 Simulation Studies

4.1 Simulation Study 1

To evaluate the performance of the proposed estimator, we conducted several simulation studies and practical examples.

In the simulation studies, both Bayesian and non-Bayesian estimators are compared for different sample sizes $n \in \{10, 20, 50\}$ and window sizes $m \in \{2, 3, 4, 7\}$. The distributions considered are:

- Exponential Exp(1) with true extropy -0.25,
- Normal $N(0, 1)$ with true extropy $-\frac{1}{2\sqrt{\pi}} \approx -0.141$.

For each case, 5000 samples were generated and the extropy was estimated for each sample. The average estimate (Est) and the mean squared error (MSE) were then calculated as:

$$MSE = \frac{1}{5000} \sum_{i=1}^{5000} (\text{Estimate}_i - \text{True Value})^2$$

All computations were carried out in R. The results are reported in Tables 1–2.

According to the simulation results, we conclude that the new approximation is more accurate in estimating extropy.

Table 1: Results for Exponential Distribution with Mean 1

$JQ_{m,n}$	$J_{m,N,a}\|X$	m	n
Est(MSE)	Est(MSE)		
$-0.2117(0.0097)$	$-0.5548(13.8784)$	3	10
$-0.1608(0.0101)$	$-0.3703(0.7433)$	4	20
$-0.1868(0.0050)$	$-0.2331(0.2397)$	7	50

True value: -0.25

Table 2: Results for Normal Distribution $N(0,1)$

$JQ_{m,n}$	$J_{m,N,a}\|X$	m	n
Est(MSE)	Est(MSE)		
$-0.1519(0.0023)$	$-0.2942(10.6124)$	3	10
$-0.1098(0.0015)$	$-0.1997(0.3497)$	4	20
$-0.1234(0.0005)$	$-0.1197(0.0694)$	7	50

True value: ≈ -0.1410

Analysis of Table 3 Table 3 shows extropy estimates for different base distributions H and two values of the concentration parameter a.

When $a = 0.05$, the estimates are similar across different H, suggesting low sensitivity to the prior. However, with $a = 5$, the estimates vary more, indicating that the prior has a stronger influence.

Thus, to obtain estimates that rely more on the observed data, a smaller value of a is recommended.

Table 3: Effect of Parameters a and H on the Estimation

Estimate: $a = 5$	**Estimate:** $a = 0.05$	**Base Distribution** H
-0.3520	-0.1706	$N(0,1)$
-0.2138	-0.1688	$N(3,9)$
-0.2886	-0.1654	t_1
-0.3715	-0.1816	Exp(1)

4.2 Simulation Study 2

For computing relative belief ratio, we must compute prior and posterior densities of $D = d(f||f_0)$, which is not available. So, we need to approximate them via simulations. A detailed computational algorithm Al-Labadi and Evans (2018) for assessing $H_0 : F = F_0$ as follows:

Algorithm A (Computing Relative Belief Ratio Algorithm for Goodness of Fit Tests):

1. Generate a sample of size r_1 from P_N, where P_N is an approximate for $DP(\alpha, F_0)$, then compute $d(\text{pr}) = \hat{d}$ as described in the text.

2. Generate a sample of size r_2 from the approximate of $P_{N|x}$ $(DP(a+n, H_x))$, then compute $d(\text{po}) = \hat{d}|x$

3. For a fixed positive number M, let $\hat{F}_D$ denote the empirical cdf of D from the prior sample. For $i = 0, \ldots, M$, $\hat{d}_{i/M(\text{pr})}$ is the estimate of the i/M-th prior quantile of D, where $\hat{d}_0(\text{pr}) = 0$ and $\hat{d}_1(\text{pr})$ is the largest value of $d(\text{pr})$.

4. Let $\hat{F}_D(\cdot|x)$ denote the empirical cdf of D based on the posterior sample. For $d \in [\hat{d}_{i/M(\text{pr})}, \hat{d}_{(i+1)/M(\text{pr})}]$, estimate $RBD(d|x)$ as the ratio of the posterior and prior cdf differences in this interval:

$$R\hat{B}D(d|x) = M\left(\hat{F}_D(\hat{d}_{(i+1)/M(\text{pr})}|x) - \hat{F}_D(\hat{d}_{i/M(\text{pr})}|x)\right).$$

5. Estimate $RBD(0|x)$ as:

$$R\hat{B}D(0|x) = M\hat{F}_D(\hat{d}_{i_0/M(\text{pr})}|x),$$

where i_0 is chosen so that i_0/M is not too small (typically $i_0/M \approx 0.05$).

6. Finally, estimate the strength $DPD(RBD(d|x) \leq RBD(0|x)|x)$ using the following summation:

$$\sum_{i \geq i_0 : R\hat{B}D(\hat{d}_{i/M(\text{pr})}|x) \leq R\hat{B}D(0|x)} \left(\hat{F}_D(\hat{d}_{(i+1)/M(\text{pr})}|x) - \hat{F}_D(\hat{d}_{i/M(\text{pr})}|x)\right).$$

In the following, the proposed method is evaluated using an example with parameters set in Algorithm A: $r = 5000$, $N = 500$, $M = 20$, and $i_0 = 1$. The sensitivity to the parameter a is examined, with selected values of a reported in the table. A sample of size $n = 25$ is drawn from the standard normal distribution, and the method is used to test the null hypothesis $H_0 : F = F_0$, as detailed in Table 4. The results for the Relative Belief Ratio (RB) and Strength are shown. When H_0 is true, we expect $RB > 1$ and Strength near 1; when H_0 is false, $RB < 1$ and Strength near 0. The proposed method performs well in this example.

Table 4: Relative Belief Ratios and Strength for different values of a and F_0 in Example

Strength	RB	a	F_0
0.7623	7.5284	1	$N(0,1)$
0.8670	6.4202	5	$N(0,1)$
0.7027	9.2031	10	$N(0,1)$
0	0	1	$N(2,1)$
0	0	5	$N(2,1)$
0	0	10	$N(2,1)$
0	3.3320	1	Exp(1)
0	3.1230	5	Exp(1)
0.0142	3.0000	10	Exp(1)

5 Conclusion

In this paper, we introduced a Bayesian nonparametric goodness of fit test based on extropy. The proposed method demonstrated superior flexibility, accuracy, and interpretability compared to traditional tests, particularly for small samples and complex deviations.

References

Al-Labadi, L., & Evans, M. (2018). Prior-based model checking. *Canadian Journal of Statistics, 46*, 380–398.

Evans, M. (2015). *Measuring Statistical Evidence Using Relative Belief*. Chapman and Hall/CRC.

Ferguson, T. S. (1973). A Bayesian analysis of some nonparametric problems. *The Annals of Statistics, 1*(2), 209–230.

Grzegorzewski, P., & Wieczorkowski, R. (1999). Entropy-based goodness-of-fit test for exponentiality. *Communications in Statistics - Theory and Methods, 28*, 1183–1202.

Lad, F., Sanfilippo, G., & Agro, G. (2015). Extropy: Complementary dual of entropy. *Statistical Science, 30*(1), 40–58.

Qiu, G., & Jia, K. (2018). Extropy estimators with applications in testing uniformity. *Journal of Nonparametric Statistics, 30*, 182–196.