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Smooth Transition Markov Switching bivariate copulas

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Abstract

Multivariate GARCH Models are widely used to model financial time series. Significant asymmetry both in the marginal distributions and in the dependence structures of financial returns are documented in during times of market stress. Copulas are widely used as a tool to separate asymmetry in the marginals from asymmetry in the dependencies. Smooth transition hyperbolic GARCH models are considered to provide a continuum of regimes in marginals. To capture the asymmetry in the dependence structure, we rely on a Markov switching copula distribution. To generate different tail dependencies for each pair of variables we use regular vine (R-vine) copulas. The parameters of the marginal are estimated separately from the copulas distributions. Our estimation procedure employs a two-step maximization approach grounded in the EM algorithm. Empirical analysis of real data provides additional validation for the proposed model.

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1 Introduction

Multivariate GARCH (MGARCH) Models are widely used as a tool to modelling conditional dependence for multivariate financial time series such as the constant conditional correlation (CCC)-GARCH models in Bollerslev (1990), the VEC model of Bollerslev (1988), the BEKK model of Engle (1995) and the dynamic correlation (DCC)-GARCH models of Engle (2002). Also the assumptions on the distribution of multivariate return series are often limited to normal distribution, t distribution or other elliptical distributions.

In the literature on financial time series, different behaviours during times of market stress has long been recognized, and return series tend to exhibit asymmetric dependence in these situations. This asymmetry means that in times of crisis, returns tend to be more dependent than they are in good times where the shape and level of correlations change over time. They are typically higher in more volatile periods and are different for positive and negative returns.

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A class of models addressing this, are Markov switching (MS), also called regime switching or hidden Markov models Hamilton (1989). There, the behaviour of a financial time series exhibits two or more distinct regimes, which correspond to different states of the economy. The regime is present at a particular point of time is governed by an underlying hidden Markov process Stöber and Czado (2014).

Patton (2004) finds significant asymmetry both in the marginal distributions and in the dependence structures of financial returns. He finds that knowledge of asymmetric dependence leads to significant gains for an investor. Copulas allow to separate asymmetry in the marginals from asymmetry in . They provide more flexibility in modelling the marginals from the dependence structure.

Apart from the multivariate Gaussian and t copulas, the set of higher-dimensional copulas are available in the literature is rather limited. Further, characteristics of financial data such as the asymmetric dependence and tail dependence must be incorporated appropriately. Choice of copulas is usually thought to be reduced to the Gaussian or the t-Student . Both of these copulas are useful only for capturing linear dependence. The Gaussian copula suffers from the drawback that it lacks tail dependence, and the multivariate t-Student copula is too restrictive in the sense that, while it can generate different tail dependences for each pair of variables (since the tail dependence is a function of the correlation and the degrees of freedom), it restricts the upper and lower tail dependences for each pair to be the same. For real life data,the dependence structure among pairs of variables may vary substantially, ranging from independence to complex nonlinear dependence. No existing family of multivariate copulas can handle such a wide range of dependence for each pair of variables.

An alternative way to make multivariate copulas is regular vine (R-vine) copulas Bedford and Cook (2002). The aim is to combine different types of margins and corresponding pair-copulas inside a multivariate distribution. R-vine distributions are built up hierarchically from bivariate copulas as the building blocks called "pair-copula construction" (PCC). In other words, the basic modelling scheme is to decompose the multivariate distribution into a cascade of bivariate pair-copulas applied on the random variables and their conditional or unconditional distribution functions. The individual pair-copulas may belong to any parametric or non-parametric family. Therefore, all types and strengths of dependence maybe accommodated in the model.

Our contribution is to introduce a smooth transition Markov switching R-vine (STMS-RV) copula model. In STMS-RV, the margins follow a hyperblic GARCH model witch has a smooth transition structure Valizadeh and *et al.* (2021). The smooth transition behaviour are imposed based on a logistic function of the last observation and provide a much better fitting for the volatilities. The smooth transition models are an extension of the two-regime models which allow a continuum of regimes between two extreme regimes associated with the extreme value of the transition function Mohammadi Basatini and Rezakhah (2019). In our proposed model the dependence structures are Markov switching copulas. the Parameters are estimated via a two-step estimation procedure. Chollete and *et al.* (2009) modelled the international financial returns with regime-switching canonical vine copula and they found that ignoring asymmetric dependence and regime switching selection leads to significant costs for an investor. Stöber and Czado (2014) studied sudden switches in dependence structures using Markov switching regular vine copulas.

2 The Smooth Transition Markov Swithing Copula

2.1 Regular vine distribution

An R-vine v on d variables, which consists of a sequence of connected trees $T_1, \ldots, T_{d-1}$, with N_i and edges E_i, $1 \leq i \leq d-1$, defind as

1. T_1 is a tree with nodes $N_1 = \{1, \ldots, d\}$ and edges E_1.

2. For $i \geq 2$, T_i is a tree with nodes $N_i = E_{i-1}$ and edges E_i.
3. If two nodes in T_{i+1} are joined by an edge, the corresponding edges in T_i must share a common node.

The main advantage of PCC is that it is not require that all the associated bivariate pair-copula belong to the same family. Two main types of PCCs have been proposed in the literature; canonical vines (C-vine) and D-vines. Here we concentrate on D-vines. Let $\mathbf{X} = (X_1, \ldots, X_d)$ be a random vector with marginal densities $f_1, \ldots, f_d$, respectively. R-vine structure associates to each edge $u(e), v(e) \mid D(e)$ in E_i, $1 \leq i \leq d-1$ a bivariate copula density $c_{u(e),v(e)|D(e)}$. Where $u(e)$ and $v(e)$ are the conditioned set and $D(e)$ is the conditioning set. The joint density of $\mathbf{X}$ is

$$f_{1,\ldots,d}(x_1, \ldots, x_d) = \prod_{i=1}^{i=d-1} \prod_{e \in E_i} c_{u(e),v(e)|D(e)}(F_{u(e)|D(e)}(x_{u(e)} \mid \mathbf{X}_{D(e)}), F_{v(e)|D(e)}(x_{v(e)} \mid \mathbf{X}_{D(e)})) \prod_{i=1}^{d} f_i(x_i). \quad (1)$$

Where $\mathbf{X}_{D(e)}$ denotes the sub-vector of $\mathbf{X}$ determined by the set of indices $D(e)$ and "f" and "F" denote probability density function and cumulative distribution function respectively. The PCCs are a cascade of $d(d-1)/2$ bivariate copula densities, of which the first $d-1$ unconditional and the rest conditional. As an example, consider $d = 3$ where $\mathbf{X} = (X_1, X_2, X_3)$. Following the previous construction, the PCC expansion becomes

$$\begin{aligned} f(x_1, x_2, x_3) = f(x_1) f(x_2) f(x_3) \\ \times c_{1,2}(F(x_1), F(x_2)) c_{2,3}(F(x_2), F(x_3)) \\ c_{1,3|2}(F(x_1 \mid x_2), F(x_3 \mid x_2)). \end{aligned}$$

Fig. 1 shows the tree representation of a D-vine, in five dimensions. It consists of four trees arranged in four levels. An edge of a tree corresponds to a pair-copula density denoted by the edge label. For example, the edge 13|2 denotes the pair-copula $c_{13|2}$.The whole structure has $d(d-1)/2$edges.

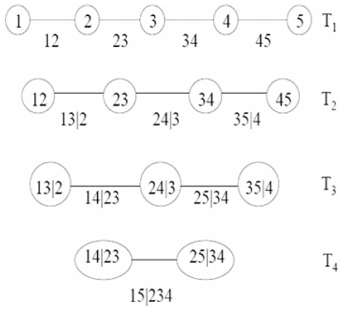


Figure 1: D-vine representation of 5 variables, 4 trees and 10 edges. Each edge is associated with pair-copula.

2.2 Markov switching copula

To model the dependence we use MS copula with two regimes, characterized by different levels or shapes of dependence. Let $\mathbf{X}_t = (X_{1,t}, \ldots, X_{d,t}), t = 1, \ldots, T$ denote a multivariate financial time series. The dependency among $\mathbf{X_t}$ depends on a hidden latent Markov chain Z_t with values $k = 1, 2$. These are called regimes and represent the different states of the economy. It is assumed that Z_t is a homogeneous first order Markov chain with transition probability matrix

$$\mathbf{P} = \begin{bmatrix} p_{11} & 1 - p_{11} \\ 1 - p_{22} & p_{22} \end{bmatrix},$$

where $P_{kk} = P(Z_t = k \mid Z_{t-1} = k)$, $k = 1, 2$ and with stationary probabilities $\pi = [\pi_1, \pi_2]$(Mohammadi Basatini and Rezakhah (2020)). In our model the regime only affects the dependence structure and the marginal distributions do not depend on regime. So the Markov-switching density of $\mathbf{X}_t$ conditional on being in regime k, is

$$f(\mathbf{X_t} \mid \mathbf{X}^{t-1}, z_t = k) = \prod_{i=1}^{i=d-1} \prod_{e \in E_i} c^{(k)}_{u(e),v(e)|D(e)}(F_{u(e)|D(e)}(x_{u(e),t} \mid \mathbf{X}_{D(e)}), F_{v(e)|D(e)}(x_{v(e),t} \mid \mathbf{X}_{D(e)}); \theta^{(k)}_c) \cdot \prod_{i=1}^{d} f_i(x_{i,t}; \theta_{m,i}), \tag{2}$$

where $\mathbf{X}^t = (\mathbf{X}'_1, \ldots, \mathbf{X}'_t)'$ be the available information up to t. Also $\theta^{(k)}_c$ denotes the parameters of the copula in regime k and $\theta_{m,i}$ denotes the parameters of the marginal distributions.

2.3 Smooth transition hyperblic GARCH model

In STMS-RV we allow the marginal distribution of each one of the returns follows a smooth transition hyperbolic GARCH (ST-HGARCH) model (Valizadeh and *et al.* , 2021).
Specifically $x_{i,t}$ follows the ST-HGARCH(q, r, p) model as

$$x_{i,t} = \epsilon_{i,t}\sqrt{h_{i,t}} \qquad i = 1, \ldots d$$
$$h_{i,t} = \frac{\gamma_i}{\beta_i(1)} + w_{i,t}[1 - \frac{\delta_i(B)}{\beta_i(B)}(1-B)^r]x^2_{i,t}, \tag{3}$$

where $\epsilon_{i,t}$ are identically and independently random variables with mean 0 and variance 1, $\gamma_i > 0$, B is the back-shift operator, $\beta_i(y) = 1 - \sum_{j=1}^{p} \beta_{i,j} y^j$, $\delta_i(y) = 1 - \sum_{j=1}^{q} \delta_{i,j} y^j$ and p, q are known positive integers; $(1-B)^r = 1 - \Sigma_{j=1}^{\infty} g_j B^j$ where $g_j = \dfrac{r\Gamma(j-r)}{\Gamma(1-r)\Gamma(j+1)}$ in which $0 < r < 1$. Also $h_{i,t}$ is the conditional variance of $x_{i,t}$ as, $\mathrm{Var}(x_{i,t}|x_i^{t-1}) = h_{i,t}$, where $x_i^t = (x_{i,1}, \ldots x_{i,t})$ denotes the history of the variable i. The parameter $w_{i,t}$ is called the amplitude parameter that determines the magnitude of variations in the conditional variance.

Here we consider $w_{i,t}$ as a logistic function defined as

$$w_{i,t} = \frac{\exp(\eta_i x^2_{i,t-1})}{1 + \exp(\eta x^2_{i,t-1})}. \tag{4}$$

It is clear that $w_{i,t}$ bounded between 0 and 1; $\eta_i > 0$ is called the smoothness parameter which determines the speed of transition between high and low volatilities. So the amplitude changes with the size of the last observation and hence the magnitude of the last shock causes smooth changes of the conditional variance.

3 Estimation

The proposed model has the advantage that the marginal parameters and the copula parameters are estimated separately in a two-step procedure. We use EM algorithm to estimate the RS copula parameters.

3.1 Two-Step Estimation

In STMS-RV model the marginal are not regime–switching, so we use this advantage to separate the estimation into two steps. Let $\mathbf{X} = (\mathbf{X}'_1, \ldots, \mathbf{X}'_T)'$ denote all the data, the total loglikelihood is given by

$$L(\mathbf{X}; \theta_m, \theta_c) = \sum_{t=1}^{T} \log f(\mathbf{X}_t \mid \mathbf{X^{t-1}}; \theta_m; \theta_c), \tag{5}$$

where θ_m and θ_c denote the parameters of the marginals and of the RS copula, respectively. The likelihood decompose in two part; L_m that contains the marginal densities and L_c which contains the dependence structure:

$$\begin{aligned} L(\mathbf{X};\theta_m,\theta_c) &= L_m(\mathbf{X};\theta_m) + L_c(\mathbf{X};\theta_m,\theta_c), \\ L_m(\mathbf{X};\theta_m) &= \sum_{t=1}^{T}\sum_{i=1}^{d} \log f_i(x_i \mid x_i^{t-1};\theta_{m,i}), \\ L_c(\mathbf{X};\theta_m,\theta_c) &= \sum_{t=1}^{T} \log c(F_1(x_{1,t} \mid x_1^{t-1};\theta_{m,1}),\dots,F_d(x_{d,t} \mid x_d^{t-1},\theta_{m,d});\theta_c), \end{aligned} \tag{6}$$

where $x_i^t = (x_{i,1},\dots x_{i,t})$ denotes the history of the variable i. For simplicity we show the dependency structure by $c(F_1(x_{1,t} \mid x_1^{t-1};\theta_{m,1}),\dots,F_d(x_{d,t} \mid x_d^{t-1},\theta_{m,d});\theta_c)$ which follows R-vine copulas. The likelihood of the marginal models, L_m is a function of the parameter vector $\theta_m = (\theta_{m,1},\cdots,\theta_{m,d})$, that collects the parameters of each one of the d marginal densities f_i. The copula likelihood, L_c, depends directly on the vector $\theta_c = (\theta_c^{(1)},\theta_c^{(2)},p_{11},p_{22})$. This vector collects the copula parameters over both regimes as well as the parameters of the Markov transition probability matrix. It also depends indirectly on the parameters of the marginal densities, through the distribution function F_i, because F_i transforms observations into uniform $[0,1]$ variables that are the input of the copula. The function "c"denotes the density of the RS copula. There are a large number of parameters in STMS-RV model, that should be estimated. In addition, the non-linearity in the copula increases the difficulty of the estimation. So we rely on a two-step maximization instead of full one-step maximization. In the first step, we estimate the marginal parameters:

$$\hat{\theta}_m = arg\max_{\theta_m} L_m(\mathbf{X};\theta_m).$$

This is straightforward, as it does not depend on the RS and, in addition, it can be simplified further by noting that we can actually estimate each ST-HGARCH model separately:

$$\hat{\theta}_{m,i} = arg\max_{\theta_{m,i}} \sum_{t=1}^{T} \log f_i(x_{i,t} \mid x_i^{t-1};\theta_{m,i}).$$

Let vector, $\hat{\theta}_m = (\hat{\theta}_{m,i},...,\hat{\theta}_{m,d})$ denotes marginal estimated parameters, in the second step we use $\hat{\theta}_m$ to estimate the parameters of the regime switching copula:

$$\hat{\theta}_c = arg\max_{\theta_c} L_c(\mathbf{X};\hat{\theta}_m,\theta_c).$$

3.2 EM Algorithm

The parameter of the RS copula are estimated conditional on the estimation of the marginal models Chollete and *et al.* (2009). Knowing the fact that the Markov chain Z_t is hidden, we can apply the filter of Hamilton (1989). explicitly, the filtered system follows

$$\hat{\xi}_{t|t} = \frac{\hat{\xi}_{t|t-1} \bigodot \eta_t}{\mathbf{1}'(\hat{\xi}_{t|t-1} \bigodot \eta_t)'} \tag{7}$$

$$\hat{\xi}_{t+1|t} = \mathbf{P}'\hat{\xi_{t|t}}, \tag{8}$$

$$\eta_t = \begin{pmatrix} c^{(1)}(F_1(x_{1,t} \mid x_1^{t-1};\theta_{m,1}),\dots,F_d(x_{d,t} \mid x_d^{t-1});\theta_c^{(1)}) \\ c^{(2)}(F_1(x_{1,t} \mid x_1^{t-1};\theta_{m,1}),\dots,F_d(x_{d,t} \mid x_d^{t-1});\theta_c^{(2)}) \end{pmatrix} \tag{9}$$

and we obtain all probabilities which are required to evaluate $L_c(\mathbf{X};\hat{\theta}_m,\theta_c)$ recursively. Where $\hat{\xi}_{t|t} = \Big(P(Z_t = 1 \mid \mathbf{X^t},\theta_\mathbf{m},\theta_\mathbf{c}), P(Z_t = 2 \mid \mathbf{X^t},\theta_\mathbf{m},\theta_\mathbf{c})\Big)$ is the (2×1) vector containing the probability of being in each regime at time t, conditional on the observations up to time t; $\mathbf{1}$ is a (2×1) vector of 1s; and $\bigodot$ denotes componentwise multiplication of two vectors. The(2×1)

Table 1: Descriptive statistics for the Oil daily log returns.

Series	Mean	Std.devd	Minimum	Maximum	Skewness	Kurtosis
CO	0	0.037	-0.281	0.426	1.334	37.335
CG	-0.011	2.997	-29.973	22.2358	-1.224	22.441
HO	0.014	2.470	-18.444	14.891	-0.374	8.293

vector $\hat{\xi}_{t+1|t}$ gives these probabilities at time $t+1$ conditional on observations up to time t. The vector η_t contains the copula density at time t, conditional on being in each one of the two regimes. Equation (7) corresponds to a Bayesian updating of the probability of being in either regime given present time observations (η_2). Equation (8) consists in doing one forward iteration of the Markov chain. Iterating over both equations from a given starting value $\hat{\xi}_{1|0}$ and parameter values $\theta_c^{(1)}$ and $\theta_c^{(2)}$ of the copula in each of the two regimes and p_{11} and p_{22} of the Markov chain, one obtains the value of the likelihood:

$$L_c(\mathbf{X}; \theta_m, \theta_c) = \sum_{t=1}^{T} \log(\mathbf{1}'(\hat{\xi}_{t|t-1} \bigodot \eta_t)). \tag{10}$$

4 Real data

We study a financial dataset, consists of a trivariate time series of oil and its products prices of n = 1763 observations. Oil and its products prices are central to global economic activity, create significant impacts on current and future economic prosperity. We consider three time series of spot price returns: crude oil (CO), conventional gasoline (CG) and heating oil (HO). We use daily observations from 10 January 2014 to 21 January 2021 from the Energy Information Administration (EIA) of the US Department of Energy; we have 1763 valid observations. We have considered the percentage log returns $r_t = \log(P_t/P_{t-1}) * 100$, where P_t shows the price at t. Some descriptive statistics of the data are listed in Table 1.

All series show very clear signs of non-normality with negative skewness for CG and HO and positive skewness for CO. Further evidence of non-normality is given by the fact that all series have a kurtosis that is well above 3. The estimates of each of the univariate t Student ST-HGARCH(1,d,1) models are presented in Table 2. The degrees-of-freedom parameters ν of all series is around 8, which corresponds to the tails of the conditional distribution that are somewhat fatter than those of the normal distribution. Table 3 shows the Kendall's τ of the data. It is obvious the strongest pair is between CO and HO with $\tau = 0.551$. Fig. 2 shows the pairwise scatter plots of the data. Obviously all pairs show symmetric low and up tail dependencies, so we considered t Student copulas as bivariate building blocks. We compare a four-dimensional pair-copula decomposition with t Student copula for all pairs with the Markov switching pair-copula. According to Kendall's τ results, in the first tree, the first edge is between CO and HO; the second edge is between HO and CG. In the second tree there is one edge between CO and CG conditional on HO. Table 4 and 5 present the results. The results of Table 5 indicate that we have a high and a low dependence regime. The copula correlation coefficient in the more dependent regime, say regime 1 is higher for all pairs of variables, which means that the Oil and its products prices more dependent when the economy is in that regime. This regime is characterized by large correlations.

Table 2: The results of t Student ST-HGARCH(1,r,1) marginal models of the Oil daily log- returns.

Series	γ	β	δ	r	η	ν
CO	0.102	0.102	0.403	0.630	0.129	8.601
CG	0.552	0.711	0.251	0.633	0.298	7.720
HO	0.359	0.734	0.239	0.672	0.299	7.879

Table 3: Kendall's τ coefficients of the Oil daily log- returns.

(CO-CG)	(CO-HO)	(CG-HO)
0.476	0.551	0.533

Table 4: The results of the four-dimensional pair-copula decomposition with t Studen copula for all pairs of the Oil daily log- returns.

Tree	Edge	Coefficient	ν	τ
1	CO,CG	0.75	4.013	0.54
	HO,CG	0.74	4.41	0.53
2	CO,CG|HO	0.28	6.44	0.18

Discussion and Results

We present additional empirical evidence regarding asymmetric dependence in financial returns through the estimation of a STMS-RV copula model specifically for the prices of oil and its derivatives. The ST-HGARCH model is employed to capture the marginal distributions, while MS R-vine copulas are utilized to represent the dependence structure. The ST-HGARCH model offers a superior characterization of dynamic volatilities, and the MS R-vine copulas provide enhanced flexibility in modelling various shapes and levels of dependencies.

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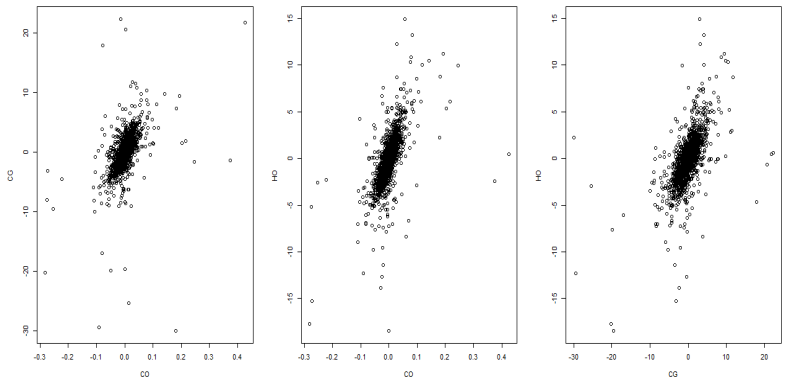


Figure 2: Pairwise scatter plots between crude oil (CO), conventional gasoline (CG) and heating oil (HO) .

Table 5: The results of the MS-copula from t Studen D-vine copulas for all pairs of the Oil daily log- returns.

		Regime 1		Regime 2	
Tree	Edge	Coefficient	τ	Coefficient	τ
1	CO,CG	0.92	0.743	0.59	0.40
	HO,CG	0.80	0.59	0.48	0.32
2	CO,CG|HO	0.71	0.50	0.40	.27

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