پانزدهمین سمینار احتمال
و فرآیندهای تصادفی
۸ و ۹ شهریور ۱۴۰۴
دانشگاه کردستان

15th Seminar On Probability and Stochastic Processes
August 30-31, 2025
University of Kurdistan

انجمن آمار ایران

Stochastic Comparisons of Series Systems Consisting of Dependent and Heterogeneous Exponential Components under Archimedean Copula

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Abstract

This paper investigates series systems comprising m dependent subsystems, each containing heterogeneous and dependent Exponential components. The dependence among components is modelled using Archimedean copulas. We establish sufficient conditions for the stochastic comparison of two such series systems when the vector of scale parameters of the first system majorizes that of the second, and when the vectors representing the number of components in each subsystem differ between the two systems. A numerical example is provided to illustrate the theoretical findings.

Keywords: Series system, Usual stochastic order, heterogeneous Exponential components.
Mathematics Subject Classification (2020): 47A55, 39B52, 34K20.

1 Introduction

In many real-world applications, including reliability engineering, lifetime populations are often heterogeneous and composed of multiple homogeneous sub-populations Shojaee and *et al.* (2021) and Shojaee and *et al.* (2022). This heterogeneity can arise from factors such as variations in production lines, differences in raw materials, or inconsistencies in work shifts and labor quality. For a detailed discussion on heterogeneous components, we refer the interested readers to Shojaee and Babanezhad (2023), Shojaee and *et al.* (2023) and Shojaee and *et al.* (2024).

This paper derives sufficient conditions for the stochastic comparison of two series systems with dependent and heterogeneous Exponential-distributed components. Specifically, we establish ordering results under the usual stochastic order when (i) the vector of scale parameters of the first system majorizes that of the second, and (ii) the vector representing the number of components in each subsystem also follows a majorization relationship.

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The rest of the paper is organized as follows. Section 2 provides some basic definitions and lemma. Section 3 is devoted to the usual stochastic order of series systems. Section 4 concludes the paper.

2 Some basic concepts

Let the random variables X and Y have survival functions $\bar{F}$ and $\bar{G}$, respectively.

Definition 2.1. *It is said to be the random variable X is smaller than Y in the usual stochastic order if $\bar{F}(x) \leq \bar{G}(x)$ for all x, and denoted by $X \leq_{st} Y$ or $\bar{F} \leq_{st} \bar{G}$.*

Definition 2.2. *(Marshall and et al. (2011)). Let $a_{(1)} \leq \cdots \leq a_{(n)}$ and $b_{(1)} \leq \cdots \leq b_{(n)}$ denote the increasing arrangements of $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$, respectively.*

(i) If $\sum_{j=1}^{i} a_{(j)} \leq \sum_{j=1}^{i} b_{(j)}$ for $i = 1, \ldots, n-1$, and $\sum_{j=1}^{n} a_{(j)} = \sum_{j=1}^{n} b_{(j)}$, then $\boldsymbol{a}$ is said to majorize $\boldsymbol{b}$ and denoted by $\boldsymbol{a} \overset{m}{\succeq} \boldsymbol{b}$.

(ii) If $\sum_{j=1}^{i} a_{(j)} \leq \sum_{j=1}^{i} b_{(j)}$ for $i = 1, \ldots, n$, then $\boldsymbol{a}$ is said to weakly supermajorize $\boldsymbol{b}$, and denoted by $\boldsymbol{a} \overset{w}{\succeq} \boldsymbol{b}$.

(iii) If $\sum_{j=i}^{n} a_{(j)} \geq \sum_{j=i}^{n} b_{(j)}$ for $i = 1, \ldots, n$, then $\boldsymbol{a}$ is said to weakly submajorize $\boldsymbol{b}$, denoted by $\boldsymbol{a} \succeq_w \boldsymbol{b}$.

Let us consider the definition of Archimedean copula from Nelsen (2007).

Definition 2.3. The copula function

$$C_{\Psi}(v_1, \ldots, v_n) = \Psi\Big(\sum_{i=1}^{n} \varphi(v_i)\Big), \qquad (v_1, \ldots, v_n) \in [0,1]^n,$$

where $\Psi : [0, +\infty) \mapsto [0,1]$ is $(n-2)$th differentiable, with $\Psi(0) = 1$ and $\lim_{t \to +\infty} \Psi(t) = 0$ is called an Archimedean survival copula with generator Ψ if $(-1)^s \Psi^s(t) \geq 0$, $s = 0, 1, \ldots, n-2$ and $(-1)^{n-2}\Psi^{n-2}(t)$ is decreasing and convex for all $t \geq 0$. In this copula $\varphi = \Psi^{-1}$ is the inverse of Ψ and is right continuous $\varphi(v) = \Psi^{-1}(v) = \sup\{t \in \mathcal{R} : \Psi(t) > v\}$.

Definition 2.4. *(Marshall and et al. (2011)). Consider a real-valued function ϕ defined on a set $\mathbb{A} \subseteq \mathbb{R}^n$. If $\boldsymbol{a} \overset{m}{\succeq} \boldsymbol{b}$ implies $\phi(\boldsymbol{a}) \geq (\leq) \phi(\boldsymbol{b})$ for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{A}$, then ϕ is Schur-convex (Schur-concave) on $\mathbb{A}$.*

Characterizations of Schur-convex (Schur-concave) functions are provided in the next lemma.

Lemma 2.5. *(Marshall and et al. (2011)). Suppose that $\phi : I^n \to \mathbb{R}$ be a real-valued, continuously differentiable function, where $I \subseteq \mathbb{R}$ is an open interval. Then, ϕ is Schur-convex (Schur-concave) on I^n if and only if ϕ is symmetric on I^n, and for all $i \neq j$ and all $\boldsymbol{x} \in I^n$, $(a_i - a_j)\left(\frac{\partial \phi}{\partial a_i}(\boldsymbol{a}) - \frac{\partial \phi}{\partial a_j}(\boldsymbol{a})\right) \geq (\leq) 0$, where $\frac{\partial \phi}{\partial a_i}$ is the partial derivative of ϕ with respect to its i-th argument.*

Lemma 2.6. *(Marshall and et al. (2011)). Let ϕ be a real-valued function defined on a set $\mathbb{A} \subseteq \mathbb{R}^n$. Then,*

(i) ϕ is increasing and Schur-convex on $\mathbb{A}$ if and only if $\boldsymbol{a} \succeq_w \boldsymbol{b}$ implies $\phi(\boldsymbol{a}) \geq \phi(\boldsymbol{b})$;

(ii) ϕ is decreasing and Schur-convex on $\mathbb{A}$ if and only if $\boldsymbol{a} \overset{w}{\succeq} \boldsymbol{b}$ implies $\phi(\boldsymbol{a}) \geq \phi(\boldsymbol{b})$.

Lemma 2.7. *(Li and Fang (2015)). Let C_{Φ} and $C_{\Phi^\star}$ are two n-dimensional Archimedean copulas. If $\varphi^\star \circ \Phi$ is super-additive, then $C_{\Phi}(\boldsymbol{v}) \leq C_{\Phi^\star}(\boldsymbol{v})$, for all $\boldsymbol{v} \in [0,1]^n$ (a function g is super-additive, if $g(p) + g(q) \leq g(p+q)$, for all p and q belong to the support of g).*

Before giving the main results of the paper, we need the following notation:

$$\mathcal{S}_n = \{(\boldsymbol{a}, \boldsymbol{b}) : a_i, b_i \geq 0 \quad (a_i - a_j)(b_i - b_j) \geq 0, \quad i, j = 1, \ldots, n\}.$$

3 Usual Stochastic Order

This section examines the usual stochastic ordering of reliability functions for two series systems, each comprising m dependent subsystems. The components within each subsystem are assumed to be dependent and heterogeneous, with their dependence structure modeled using Archimedean copulas. Here, m represents the number of subsystems in a series system, while n_j (where $j = 1, \ldots, m$) denotes the number of components in the j-th subsystem. The total number of components in the system is given by $n = \sum_{j=1}^{m} n_j$.

Theorem 3.1. *Let $W_n(m, \boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{\Psi})$ and $W_n^\star(m, \boldsymbol{n}, \boldsymbol{\gamma}, \boldsymbol{\Psi}^\star)$ are the lifetimes random variables of two series systems with the respective parameters $\boldsymbol{\lambda} = (\lambda_{1,1}, \ldots, \lambda_{1,n_1}, \ldots, \lambda_{m,1}, \ldots, \lambda_{m,n_m})$ and $\boldsymbol{\gamma} = (\gamma_{1,1}, \ldots, \gamma_{1,n_1}, \ldots, \gamma_{m,1}, \ldots, \gamma_{m,n_m})$, the common generators log-convex Φ, $j = 1, \ldots, m$, and the respective generators log-convex Ψ and $\Psi^\star$ of Archimedean copulas with reliability functions*

$$\bar{F}_{W_n(m,\boldsymbol{n},\boldsymbol{\lambda},\boldsymbol{\Psi})}(x) = \Psi\Big(\sum_{j=1}^{m} \psi\Big[\Phi\Big(\sum_{i=1}^{n_j} \varphi\big(\exp(-\lambda_{j,i}x)\big)\Big)\Big]\Big)$$

and

$$\bar{F}_{W_n^\star(m,\boldsymbol{n},\boldsymbol{\gamma},\boldsymbol{\Psi}^\star)}(x) = \Psi^\star\Big(\sum_{j=1}^{m} \psi^\star\Big[\Phi\Big(\sum_{i=1}^{n_j} \varphi\big(\exp(-\gamma_{j,i}x)\big)\Big)\Big]\Big),$$

respectively. For $(\boldsymbol{n}, \boldsymbol{\lambda}) \in \mathcal{S}_m$ and $(\boldsymbol{n}, \boldsymbol{\gamma}) \in \mathcal{S}_m$, if $\boldsymbol{\lambda} \succeq_w \boldsymbol{\gamma}$ and $\psi^\star \circ \Psi$ be super-additive, then $W_n(m, \boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{\Psi}) \leq_{st} W_n^\star(m, \boldsymbol{n}, \boldsymbol{\gamma}, \boldsymbol{\Psi}^\star)$.

Proof. We must to show that $\bar{F}_{W_n(m,\boldsymbol{n},\boldsymbol{\lambda},\boldsymbol{\Psi})}(x)$ is decreasing and Schur-concave with respect to $\boldsymbol{\lambda}$. By considering $\delta(\boldsymbol{\lambda}) = \bar{F}_{W_n(m,\boldsymbol{n},\boldsymbol{\lambda},\boldsymbol{\Psi})}(x)$, the partial derivative of $\delta(\boldsymbol{\lambda})$ with respect to $\lambda_{j,i}$ is

$$\begin{aligned}\frac{\partial\delta(\boldsymbol{\lambda})}{\partial\lambda_{j,i}} &= \Psi'\Big(\sum_{j=1}^{m} \psi\Big[\Phi\Big(\sum_{i=1}^{n_j} \varphi\big(\exp(-\lambda_{j,i}x)\big)\Big)\Big]\Big) \times \psi'\Big[\Phi\Big(\sum_{i=1}^{n_j} \varphi\big(\exp(-\lambda_{j,i}x)\big)\Big)\Big] \\ &\quad \times \Phi'\Big(\sum_{i=1}^{n_j} \varphi\big(\exp(-\lambda_{j,i}x)\big)\Big) \times \varphi'\big(\exp(-\lambda_{j,i}x^\beta)\big) \times (-x\exp(-\lambda_{j,i}x)).\end{aligned}$$

The decreasingness property of generators of Archimedean copulas yields: $\Psi' \leq 0$, $\psi' = (\Psi^{-1})' \leq 0$, $\Phi' \leq 0$ and $\varphi' = (\Phi^{-1})' \leq 0$. Thus, $\frac{\partial\delta(\boldsymbol{\lambda})}{\partial\lambda_{j,i}} \leq 0$, and $\delta(\boldsymbol{\lambda})$ is decreasing in λ. In addition, for $1 \leq j, k \leq m$ and any pair (u, s), $1 \leq u \leq n_j$ and $1 \leq s \leq n_k$

$$\begin{aligned}\Big(\frac{\partial\delta(\boldsymbol{\lambda})}{\partial\lambda_{j,u}} - \frac{\partial\delta(\boldsymbol{\lambda})}{\partial\lambda_{k,s}}\Big) &= \Psi'\Big(\sum_{j=1}^{m} \psi\Big[\Phi\Big(\sum_{i=1}^{n_j} \varphi\big(\exp(-\lambda_{j,i}x)\big)\Big)\Big]\Big)\Big(\frac{(-x\exp(-\lambda_{j,u}x))}{\exp(-\lambda_{j,u}x)} \\ &\times \psi'\Big[\Phi\Big(\sum_{u=1}^{n_j} \varphi\big(\exp(-\lambda_{j,u}x)\big)\Big)\Big] \times \Phi\Big(\sum_{u=1}^{n_j} \varphi\big(\exp(-\lambda_{j,u}x)\big)\Big) \times \varphi'\big(\exp(-\lambda_{j,u}x)\big)\end{aligned}$$

$$\times \frac{\Phi'\Big(\sum_{u=1}^{n_j}\varphi\big(\exp(-\lambda_{j,u}x)\big)\Big)}{\Phi\Big(\sum_{u=1}^{n_j}\varphi\big(\exp(-\lambda_{j,u}x)\big)\Big)}\times\exp(-\lambda_{j,u}x)-\psi'\Big[\Phi\Big(\sum_{s=1}^{n_k}\varphi_j\big(\exp(-\lambda_{k,s}x)\big)\Big)\Big]$$

$$\times\Phi\Big(\sum_{s=1}^{n_k}\varphi\big(\exp(-\lambda_{k,s}x)\big)\Big)\times\frac{\Phi'\Big(\sum_{s=1}^{n_k}\varphi\big(\exp(-\lambda_{k,s}x)\big)\Big)}{\Phi\Big(\sum_{s=1}^{n_k}\varphi\big(\exp(-\lambda_{k,s}x)\big)\Big)}\times\varphi'\big(\exp(-\lambda_{k,s}x)\big)$$

$$\times\exp(-\lambda_{k,s}x)\times\frac{(-x\exp(-\lambda_{k,s}x))}{\exp(-\lambda_{k,s}x)}\Big)$$

$$\overset{sign}{=}-\Bigg(\psi'\Big[\Phi\Big(\sum_{u=1}^{n_j}\varphi\big(\exp(-\lambda_{j,u}x)\big)\Big)\Big]\times\Phi\Big(\sum_{u=1}^{n_j}\varphi\big(\exp(-\lambda_{j,u}x)\big)\Big)\times W(j,u)\times$$

$$\varphi'\big(\exp(-\lambda_{j,u}x)\big)\times\exp(-\lambda_{j,u}x)\times K(j,u)-\psi'\Big[\Phi\Big(\sum_{s=1}^{n_k}\varphi_j\big(\exp(-\lambda_{k,s}x)\big)\Big)\Big]\times$$

$$\Phi\Big(\sum_{s=1}^{n_k}\varphi_j\big(\exp(-\lambda_{k,s}x)\big)\Big)\times W(k,s)\times\varphi'\big(\exp(-\lambda_{k,s}x)\big)\times\exp(-\lambda_{k,s}x)\times$$

$$K(k,s)\Big),$$

where $W(f,g)=\frac{\partial\ln\Phi(x)}{\partial x}=\frac{\Phi'(x)}{\Phi(x)}=\frac{\Phi'\Big(\sum_{g=1}^{n_f}\varphi\left(e^{-\lambda_{f,g}t}\right)\Big)}{\Phi\Big(\sum_{g=1}^{n_f}\varphi\left(e^{-\lambda_{f,g}t}\right)\Big)}$ and $K(f,g)=\frac{\partial\ln e^{-\lambda_{f,g}t}}{\partial\lambda_{f,g}}=\frac{(-te^{-\lambda_{f,g}t})}{e^{-\lambda_{f,g}t}}$. Now, for $\lambda_{j,u}\geq\lambda_{k,s}$, $j,k=1,\ldots,m$, $u=1,\ldots,n_j$, $s=1,\ldots,n_k$,

$$\exp(-\lambda_{j,u}x)\leq\exp(-\lambda_{k,s}x). \tag{1}$$

The decreasingness of φ yields:

$$\varphi\big(\exp(-\lambda_{j,u}x)\big)\geq\varphi\big(\exp(-\lambda_{k,s}x)\big), \tag{2}$$

and from log-convexity of Φ (or equivalently, $x\varphi'(x)$ is increasing in x), we have

$$\exp(-\lambda_{j,u}x)\varphi'\big(\exp(-\lambda_{j,u}x)\big)\leq\exp(-\lambda_{k,s}x)\varphi'\big(\exp(-\lambda_{k,s}x)\big)\leq 0. \tag{3}$$

Also,

$$0\geq K(j,u)=K(k,s). \tag{4}$$

Now, from the assumption that $(\boldsymbol{n},\boldsymbol{\lambda})\in\mathcal{S}_m$, we have $n_j\geq n_k$. Then, from relation (2) we get

$$\sum_{u=1}^{n_j}\varphi\big(\exp(-\lambda_{j,u}x^{\beta})\big)\geq\sum_{s=1}^{n_k}\varphi\big(\exp(-\lambda_{k,s}x^{\beta})\big),$$

and thus, from decreasingness property of Φ, we have

$$\Phi\Big(\sum_{u=1}^{n_j}\varphi\big(\exp(-\lambda_{j,u}x)\big)\Big)\leq\Phi\Big(\sum_{s=1}^{n_k}\varphi\big(\exp(-\lambda_{k,s}x)\big)\Big). \tag{5}$$

Since Φ is log-convex, $-\Phi$ will be log-concave. Then,

$$-W(j,u)\geq-W(k,s)\equiv K(j,u)W(j,u)\geq K(k,s)W(k,s)\geq 0 \tag{6}$$

Hence, from the log-convexity of Ψ, $x\psi'(x)$ is increasing in x, and inequality (5), we have

$$\Phi\Big(\sum_{u=1}^{n_j}\varphi\big(\exp(-\lambda_{j,u}x)\big)\Big)\psi'\Big[\Phi\Big(\sum_{u=1}^{n_j}\varphi\big(\exp(-\lambda_{j,u}x)\big)\Big)\Big]$$

$$\leq\Phi\Big(\sum_{s=1}^{n_k}\varphi\big(\exp(-\lambda_{k,s}x)\big)\Big)\psi'\Big[\Phi\Big(\sum_{s=1}^{n_k}\varphi\big(\exp(-\lambda_{k,s}x)\big)\Big)\Big]\leq 0. \tag{7}$$

$$(\lambda_{j,u}-\lambda_{k,s})\left[\frac{\partial\delta(\boldsymbol{\lambda})}{\partial\lambda_{j,u}}-\frac{\partial\delta(\boldsymbol{\lambda})}{\partial\lambda_{k,s}}\right]\leq 0.$$

Thus, using Lemma 2.5, $\delta(\boldsymbol{\lambda})$ is Schur-concave with respect to $\boldsymbol{\lambda}$. Moreover, it is easy to see that $\delta(\boldsymbol{\lambda})$ is symmetric with respect to λ. Hence, by Lemma 2.6, if $\boldsymbol{\lambda}\succeq_w\boldsymbol{\gamma}$, then

$$\Psi\Big(\sum_{j=1}^{m}\psi\Big[\Phi\Big(\sum_{i=1}^{n_j}\varphi\big(\exp(-\lambda_{j,i}x)\big)\Big)\Big]\Big)\leq\Psi\Big(\sum_{j=1}^{m}\psi\Big[\Phi\Big(\sum_{i=1}^{n_j}\varphi\big(\exp(-\gamma_{j,i}x)\big)\Big)\Big]\Big). \tag{8}$$

From assumption $\psi^\star\circ\Psi$ is super-additive together with Lemma 2.7, one can obtain

$$\Psi\Big(\sum_{j=1}^{m}\psi\Big[\Phi\Big(\sum_{i=1}^{n_j}\varphi\big(\exp(-\gamma_{j,i}x)\big)\Big)\Big]\Big)\leq\Psi^\star\Big(\sum_{j=1}^{m}\psi^\star\Big[\Phi\Big(\sum_{i=1}^{n_j}\varphi\big(\exp(-\gamma_{j,i}x)\big)\Big)\Big]\Big). \tag{9}$$

By combining inequalities (8) and (9), we get

$$\Psi\Big(\sum_{j=1}^{m}\psi\Big[\Phi\Big(\sum_{i=1}^{n_j}\varphi\big(\exp(-\lambda_{j,i}x)\big)\Big)\Big]\Big)\leq\Psi^\star\Big(\sum_{j=1}^{m}\psi^\star\Big[\Phi\Big(\sum_{i=1}^{n_j}\varphi\big(\exp(-\gamma_{j,i}x)\big)\Big)\Big]\Big).$$

That means $W_n(m,\boldsymbol{n},\boldsymbol{\lambda},\boldsymbol{\Psi})\leq_{st}W_n^\star(m,\boldsymbol{n},\boldsymbol{\gamma},\boldsymbol{\Psi}^\star)$, and this completing the proof of the theorem. □

Theorem 3.1 gives the following upper bound for the SF of the random variable $W_n(m,\boldsymbol{n},\boldsymbol{\lambda},\boldsymbol{\Psi})$.

Corollary 3.2. *Set $(\gamma_1,\ldots,\gamma_n)=(\bar\lambda,\ldots,\bar\lambda)$, where $\bar\lambda=\frac{1}{n}\sum_{j=1}^{m}\sum_{i=1}^{n_j}\lambda_{j,i}$. One can see that $\boldsymbol{\lambda}\succeq_w\boldsymbol{\gamma}$. Let $(\boldsymbol{n},\boldsymbol{\lambda})\in\mathcal{S}_m$ and Ψ and Φ be log-convex. For the SF of the random variable $W_n(m,\boldsymbol{n},\boldsymbol{\lambda},\boldsymbol{\Psi})$ an upper bound is as follows:*

$$\bar F_{W_n(m,\boldsymbol{n},\boldsymbol{\lambda},\boldsymbol{\Psi})}(x)\leq\Psi\Big(\sum_{j=1}^{m}\psi\Big[\Phi\Big(n_j\varphi\big(\exp(-\bar\lambda x)\big)\Big)\Big]\Big).$$

Remark 3.3. *The assumption of sub-additivity or super-additivity of $\psi^\star\circ\Psi$ in Theorems 3.1 is not restrictive. Many Archimedean copulas satisfy this condition. We present some examples below. The super-additivity of $\psi^\star\circ\Psi$ is easy to verify.*

(i) Let $\Psi(u)=e^{\frac{1-e^u}{\theta_1}}$ and $\Psi^\star(u)=e^{\frac{1-e^u}{\theta_2}}$, where $0<\theta_1,\theta_2\leq 1$. Thus, $\psi^\star\circ\Psi$ is super-additive for $\theta_1\geq\theta_2$.

(ii) Let $\Psi(u)=\frac{\theta}{\log(e^\theta+u)}$ and $\Psi^\star(u)=\log(e^\theta+u)^{-\frac{1}{\theta}}$, where $\theta>1$. Thus, $\psi^\star\circ\Psi$ is super-additive.

Remark 3.4. *The assumption of log-convexity of the generators of Archimedean copulas in Theorems 3.1 is satisfied in the following generators.*

(i) The Gumbel-Hougaard copula with generator $\Psi(u)=e^{-u^{\frac{1}{\theta}}}$.

(ii) $\Psi(u)=\frac{\theta}{\log(e^\theta+u)}$, $\theta\in(0,+\infty)$.

(iii) $\Psi(u)=\log(e^\theta+u)^{-\frac{1}{\theta}}$, $\theta\in(0,+\infty)$.

(iv) The Clayton copula with generator $\Psi(u)=(1+\theta u)^{\frac{-1}{\theta}}$, $\theta\in(0,+\infty)$.

Now, let the components in each subsystem have the same SFs, $\bar F_j(x)=\exp(-\lambda_j x)$, $j=1,\ldots,m$, but the number of components in each subsystem of the first system is different from the second one. Also, suppose that the subsystems have the same dependency structure.

Theorem 3.5. *Suppose that $W_n(m,\boldsymbol{n},\boldsymbol{\lambda},\boldsymbol{\Psi})$ and $W_{n^\star}^\star(m,\boldsymbol{n}^\star,\boldsymbol{\lambda},\boldsymbol{\Psi}^\star)$ are the lifetimes random variables of two series systems with the common parameters $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_m)$, the common log-concave generators Φ, and the respective log-convex generators Ψ and $\Psi^\star$ of Archimedean copulas with SF's*

$$\bar F_{W_n(m,\boldsymbol{n},\boldsymbol{\lambda},\boldsymbol{\Psi})}(x)=\Psi\Big(\sum_{j=1}^{m}\psi\Big[\Phi\Big(n_j\varphi\big(\exp(-\lambda_j x)\big)\Big)\Big]\Big)$$

and

$$\bar{F}_{W^{\star}_{n^{\star}}(m,\boldsymbol{n}^{\star},\boldsymbol{\lambda},\boldsymbol{\Psi}^{\star})}(x)=\Psi^{\star}\Big(\sum_{j=1}^{m}\psi^{\star}\Big[\Phi\Big(n_j^{\star}\varphi\big(\exp(-\lambda_j x)\big)\Big)\Big]\Big),$$

where $\sum_{j=1}^{m}n_j=n$ *and* $\sum_{j=1}^{m}n_j^{\star}=n^{\star}$. *For* $(\boldsymbol{n},\boldsymbol{\lambda})\in\mathcal{S}_m$ *and* $(\boldsymbol{n}^{\star},\boldsymbol{\lambda})\in\mathcal{S}_m$, *if* $\boldsymbol{n}\succeq_w\boldsymbol{n}^{\star}$ *and* $\psi^{\star}\circ\Psi$ *be super-additive, then* $W_n(m,\boldsymbol{n},\boldsymbol{\lambda},\boldsymbol{\Psi})\leq_{st}W^{\star}_{n^{\star}}(m,\boldsymbol{n}^{\star},\boldsymbol{\lambda},\boldsymbol{\Psi}^{\star})$.

Proof. We must to show that $\bar{F}_{W_n(m,\boldsymbol{n},\boldsymbol{\lambda},\boldsymbol{\Psi})}(x)$ is decreasing and Schur-concave with respect to $\boldsymbol{n}$. By considering $\zeta(\boldsymbol{n})=\bar{F}_{W_n(m,\boldsymbol{n},\boldsymbol{\lambda},\boldsymbol{\Psi})}(x)$, the partial derivative of $\xi(\boldsymbol{n})$ with respect to n_j is

$$\begin{aligned}\frac{\partial\zeta(\boldsymbol{n})}{\partial n_j}&=\Psi'\Big(\sum_{j=1}^{m}\psi\Big[\Phi\Big(n_j\varphi\big(\exp(-\lambda_j x)\big)\Big)\Big]\Big)\times\psi'\Big[\Phi\Big(n_j\varphi\big(\exp(-\lambda_j x)\big)\Big)\Big]\\&\quad\times\Phi'\Big(n_j\varphi\big(\exp(-\lambda_j x)\big)\Big)\times\varphi\big(\exp(-\lambda_j x)\big).\end{aligned}$$

The decreasingness property of generators of Archimedean copulas yields: $\Psi'\leq 0$, $\psi'=(\Psi^{-1})'\leq 0$ and $\Phi'\leq 0$. Thus, $\frac{\partial\zeta(\boldsymbol{n})}{\partial n_j}\leq 0$, and $\zeta(\boldsymbol{n})$ is decreasing in n. In addition, for any pair (u,s), similar to the proof of Thereom 3.1, we have

$$\begin{aligned}&\Big[\frac{\partial\zeta(\boldsymbol{n})}{\partial n_u}-\frac{\partial\zeta(\boldsymbol{n})}{\partial n_s}\Big]\overset{sign}{=}-\Big[\psi'\Big[\Phi\Big(n_u\varphi\big(\exp(-\lambda_u x)\big)\Big)\Big]\times\Phi\Big(n_u\varphi\big(\exp(-\lambda_u x)\big)\Big)\times\\&\frac{\Phi'\Big(n_u\varphi\big(\exp(-\lambda_u x)\big)\Big)}{\Phi\Big(n_u\varphi\big(\exp(-\lambda_u x)\big)\Big)}\times\varphi\big(\exp(-\lambda_u x)\big)-\psi'\Big[\Phi\Big(n_s\varphi\big(\exp(-\lambda_s x)\big)\Big)\Big]\times\varphi\big(\exp(-\lambda_s x)\big)\\&-\psi'\Big[\Phi\Big(n_s\varphi\big(\exp(-\lambda_s x)\big)\Big)\Big]\times\Phi\Big(n_s\varphi\big(\exp(-\lambda_s x)\big)\Big)\times\frac{\Phi'\Big(n_s\varphi\big(\exp(-\lambda_s x)\big)\Big)}{\Phi\Big(n_s\varphi\big(\exp(-\lambda_s x)\big)\Big)}\\&\times\varphi\big(\exp(-\lambda_s x)\big)\Big].\end{aligned}$$

Now, for $u\leq s$ $(u>s)$ and from the assumption that $(\boldsymbol{n},\boldsymbol{\lambda})\in\mathcal{S}_m$, $n_u\leq(\geq)n_s$ and $\lambda_u\leq(\geq)\lambda_s$. Thus, $\exp(-\lambda_u x)\geq(\leq)\exp(-\lambda_s x)$. On the one hand, from the decreasing property of φ, we have

$$\varphi\big(\exp(-\lambda_u x)\big)\leq(\geq)\varphi\big(\exp(-\lambda_s x)\big),$$

and then

$$n_u\varphi\big(\exp(-\lambda_u x)\big)\leq(\geq)n_s\varphi\big(\exp(-\lambda_s x)\big).$$

On the other hand, from log-concavity and decreasingness of Φ, respectively, we get

$$0\geq\frac{\Phi'\Big(n_u\varphi\big(\exp(-\lambda_u x)\big)\Big)}{\Phi\Big(n_u\varphi\big(\exp(-\lambda_u x)\big)\Big)}\geq(\leq)\frac{\Phi'\Big(n_s\varphi\big(\exp(-\lambda_s x)\big)\Big)}{\Phi\Big(n_s\varphi\big(\exp(-\lambda_s x)\big)\Big)}(\leq 0),$$

$$\Phi\Big(n_u\varphi\big(\exp(-\lambda_u x)\big)\Big)\geq(\leq)\Phi\Big(n_s\varphi\big(\exp(-\lambda_s x)\big)\Big).$$

Since ψ is log-convex, one can obtain

$$\begin{aligned}0&\geq\Phi\Big(n_u\varphi\big(\exp(-\lambda_u x)\big)\Big)\psi'\Big[\Phi\Big(n_u\varphi\big(\exp(-\lambda_u x)\big)\Big)\Big]\\&\geq(\leq)\Phi\Big(n_s\varphi\big(\exp(-\lambda_s x)\big)\Big)\psi'\Big[\Phi\Big(n_s\varphi\big(\exp(-\lambda_s x)\big)\Big)\Big](\leq 0).\end{aligned}$$

Consequently,

$$(n_u-n_s)\Big[\frac{\partial\zeta(\boldsymbol{n})}{\partial n_u}-\frac{\partial\zeta(\boldsymbol{n})}{\partial n_s}\Big]\leq 0.$$

Thus, using Lemma 2.5, $\zeta(\boldsymbol{n})$ is Schur-concave with respect to $\boldsymbol{n}$. In addition, one can see that $\zeta(\boldsymbol{n})$ is symmetric with respect to n. Therefore, from Lemma 2.6, if $\boldsymbol{n} \succeq_w \boldsymbol{n}^\star$, then

$$\Psi\Big(\sum_{j=1}^{m}\psi\Big[\Phi\Big(n_j\varphi\big(\exp(-\lambda_j x)\big)\Big)\Big]\Big) \leq \Psi\Big(\sum_{j=1}^{m}\psi\Big[\Phi\Big(n_j^\star\varphi\big(\exp(-\lambda_j x)\big)\Big)\Big]\Big). \tag{10}$$

Similar to the proof of Theorem 3.1, from assumption $\psi^\star \circ \Psi$ is super-additive, we get

$$\Psi\Big(\sum_{j=1}^{m}\psi\Big[\Phi\Big(n_j\varphi\big(\exp(-\lambda_j x)\big)\Big)\Big]\Big) \leq \Psi^\star\Big(\sum_{j=1}^{m}\psi^\star\Big[\Phi\Big(n_j^\star\varphi\big(\exp(-\lambda_j x)\big)\Big)\Big]\Big).$$

That means $W_n(m,\boldsymbol{n},\boldsymbol{\lambda},\boldsymbol{\Psi}) \leq_{st} W^\star_{n^\star}(m,\boldsymbol{n}^\star,\boldsymbol{\lambda},\boldsymbol{\Psi}^\star)$. □

Let us consider the following example for our theoretical findings.

Example 3.6. *(a) Let* $\Psi(v) = \frac{2}{\log(e^2+v)}$ *and* $\Psi^\star(v) = \log(e^{1.8}+v)^{-\frac{1}{1.8}}$. *Clearly,* Ψ *and* $\Psi^\star$ *are log-convex, and* $\psi^\star \circ \Psi$ *is super-additive. Set* $n = 12$, $m = 3$, $\boldsymbol{n} = (4,4,4)$, *and* $\Phi(u) = \frac{1.5}{\log(e^{1.5}+u)}$. *Put the parameters of the systems*

$$\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_{12}) = (0.2, 0.26, 0.32, 0.38, 0.44, 0.5, 0.56, 0.62, 0.7, 0.75, 0.8, 0.9),$$

and

$$\boldsymbol{\gamma} = (\gamma_1,\ldots,\gamma_{12}) = (0.1, 0.14, 0.18, 0.22, 0.26, 0.3, 0.34, 0.38, 0.52, 0.56, 0.6, 0.7).$$

It is easy to see that $\boldsymbol{\lambda} \succeq_w \boldsymbol{\gamma}$. *Thus, all conditions of Theorem 3.1 are met. Figure 1 (a) depicts the plots of* $\bar{F}_{W_{12}(3,\boldsymbol{4},\boldsymbol{\lambda},\boldsymbol{\Psi})}(x)$ *and* $\bar{F}_{W^\star_{12}(3,\boldsymbol{4},\boldsymbol{\gamma},\boldsymbol{\Psi}^\star)}(x)$. *As can be seen from the plot* $W_{12}(3,\boldsymbol{4},\boldsymbol{\lambda},\boldsymbol{\Psi}) \leq_{st} W^\star_{12}(3,\boldsymbol{4},\boldsymbol{\gamma},\boldsymbol{\Psi}^\star)$.

(b) Let $\Psi(v) = e^{\frac{1-e^v}{0.75}}$ *and* $\Psi^\star(v) = e^{\frac{1-e^v}{0.55}}$. *Clearly,* $\psi^\star \circ \Psi$ *is super-additive. Set* $n = 9$, $n^\star = 7$, $m = 3$, $\boldsymbol{n} = (n_1,n_2,n_3) = (4,3,2)$, $\boldsymbol{n}^\star = (n_1^\star,n_2^\star,n_3^\star) = (3,2,2)$, $\Phi(u) = e^{\frac{1-e^u}{0.8}}$, *and* $\boldsymbol{\lambda} = (\lambda_1,\lambda_2,\lambda_3) = (0.5, 0.4, 0.3)$. *It is easy to see that* $\boldsymbol{n} \succeq_w \boldsymbol{n}^\star$, $(\boldsymbol{n},\boldsymbol{\lambda}) \in \mathcal{T}_3$ *and* $(\boldsymbol{n}^\star,\boldsymbol{\lambda}) \in \mathcal{T}_3$. *Thus, all conditions of Theorem 3.5 are met. Figure 1 (b) depicts the plots of* $\bar{F}_{W_9(3,\boldsymbol{n},\boldsymbol{\lambda},\boldsymbol{\Psi})}(x)$ *and* $\bar{F}_{W^\star_7(3,\boldsymbol{n}^\star,\boldsymbol{\lambda},\boldsymbol{\Psi}^\star)}(x)$, *which indicates that* $W_9(3,\boldsymbol{n},\boldsymbol{\lambda},\boldsymbol{\Psi}) \leq_{st} W^\star_7(3,\boldsymbol{n}^\star,\boldsymbol{\lambda},\boldsymbol{\Psi}^\star)$. *This coincides with the result of Theorem 3.5.*

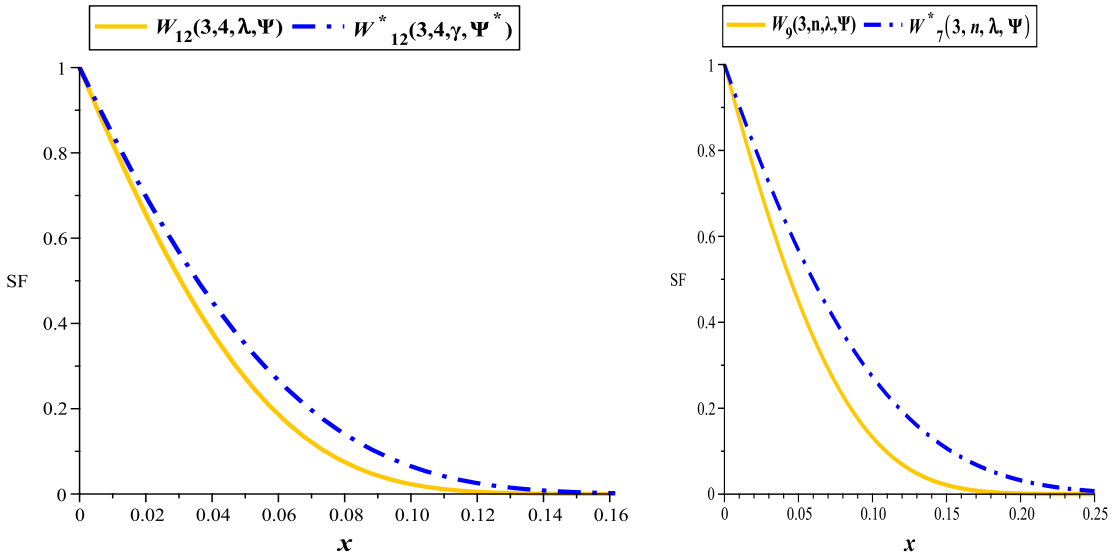


Figure 1: (a): $\bar{F}_{W_{12}(3,\boldsymbol{4},\boldsymbol{\lambda},\boldsymbol{\Psi})}(x)$ (solid) and $\bar{F}_{W^\star_{12}(3,\boldsymbol{4},\boldsymbol{\gamma},\boldsymbol{\Psi}^\star)}(x)$ (dash dot); and (b): $\bar{F}_{W_9(3,\boldsymbol{n},\boldsymbol{\lambda},\boldsymbol{\Psi})}(x)$ (solid) and $\bar{F}_{W^\star_7(3,\boldsymbol{n}^\star,\boldsymbol{\lambda},\boldsymbol{\Psi}^\star)}(x)$ (dash dot) in Example 3.6.

4 Conclusions

This paper has analyzed series systems with dependent and heterogeneous Exponential components, where dependence is modeled via Archimedean copulas. Our results demonstrate that when the first systems vector of subsystem component counts and scale parameters majorize those of the second system, the first system is less reliable than the second under the usual stochastic order. These findings provide a theoretical foundation for comparing system reliability in scenarios involving complex dependencies and heterogeneity, which are common in real-world applications such as manufacturing, telecommunications, and power grids.

Acknowledgment

Any acknowledgments should appear after the conclusion and before references. References must be cited in alphabetical order.

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