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Nonlinear Time Series Model Selection Test based on the Modified Maximum Likelihood Method

Abstract

We considered the symmetric and asymmetric types of the smooth transition autoregressive model which are suitable for modelling economic and financial time series. We derived modified maximum likelihood estimators of parameters and computed asymptotic distribution of modified maximum likelihood estimators. So that, we can derive asymptotic distribution of Vuong's test statistic for stationary smooth transition autoregressive models. A set of simulation results also leads strong support to the results presented in the paper. Using simulation, it shows that Vuong's test work to select the optimal stationary smooth transition autoregressive models.

Keywords: Model selection, Smooth transition autoregressive model, Vuong's test.
Mathematics Subject Classification (2020): 60Fxx, 62M10, 91B84.

1 Introduction

Often, in data analysis we seek to determine the underlying model or rule that is capable of describing the data. The objective of model selection is to select a model that is closest to true model. Cox (1962) and Vuong (1989) proposed model selection test for separated and non-nested models based on the likelihood ratio. Shi (2015) proposed a one-step nondegenerate test as an alternative to the classical Vuong's test. Schneider et al. (2019) applied Vuong's general approach of model selection to the comparison of nested and non-nested dimensional and multidimensional item response theory models. Liao and Shi (2020) proposed a new model selection test for the statistical comparison of semi/non-parametric models based on a general quasi-likelihood ratio criterion.

Nonlinear time series models have recently found widespread application in analysis of economics and finance time series. Chan and Tong (1986) used the cumulative function of the standard Normal variable as the transition function and introduced the smooth transition autoregressive, STAR, model as nonlinear time series model. The logistic autoregressive model and exponential autoregressive model are most popular specifications of the STAR model. Logistic smooth transition autoregressive model, LSTAR, is characterized by the asymmetric properties which make it suitable for modelling specific economics and finance time series, see Yaya and Shitu (2016). Dijk et al. (2002) shown that the convergence of the maximum likelihood estimator for STAR models is sensitive to initial values. Lopes and Salazar (2006)

proposed Bayesian approach to the logistic smooth transition autoregressive model based on the novel reversible jump markov chain Monte Carlo (RJMCMC) algorithm. The nature of numerical difficulties using Monte Carlo simulation is studied by Chan and Theoharakis (2011). They shown that the conventional optimization algorithms do not perform well in locating the global optimum of the associated likelihood function. Schleer (2016) studied the starting-values for estimation of vector STAR models based on a Monte Carlo method.
In this paper, we considered the smooth transition autoregressive model and estimated unknown parameters based on the modified maximum likelihood method and proposed Vuong's test as model selection test.
The rest of the paper is structured as follows: in Section 2 The unknown parameters of the smooth transition autoregressive model are estimated based on the modified maximum likelihood method and Vuong's test is proposed as model selection test. In Section 3, we study the obtained theoretical results by a simulation study.

2 Vuong's Test for Nonlinear Time Series Models

Consider the smooth transition autoregressive, STAR, model as:

$$y_t = \alpha y_{t-1} + \phi y_{t-1} G(s_t, \gamma, c) + \epsilon_t, t = 2, \cdots, T \tag{1}$$

where error term ϵ_t distributed as independently and identically with mean 0 and variance σ^2 and $G(s_t, \gamma, c)$ is a transition function. It usually is a continuous function, bounded between 0 and 1 and thus allowing for a smooth transition between regimes. It is assumed that the transition variable s_t be a lagged endogenous variable, i.e. $s_t = y_{t-d}$, see Teräsvirta (1994). The value of d is varied in order to improve nonlinearity in the system when it is not known prior to model estimation. The transition function $G(s_t, \gamma, c)$ causes the nonlinear dynamics in the model. The U-shaped exponential function:

$$G(s_t, \gamma, c) = 1 - exp\left\{-\gamma(s_t - c)^2\right\}, \gamma > 0 \tag{2}$$

and the first-order logistic function:

$$G(s_t, \gamma, c) = (1 + exp\left\{-\gamma(s_t - c)\right\})^{-1} - \frac{1}{2}, \gamma > 0 \tag{3}$$

are the most popular specifications of the transition function $G(.)$. When $\gamma \to 0$ the STAR model with exponential transition function, ESTAR, and the STAR model with logistic transition function, LSTAR, converge to an autoregressive model. For large values of γ, the exponential function converges to one for values of s_t below or above threshold parameter c.
Consider model (1), where error term ϵ_t distributed as Normal distribution, $N(0, \sigma^2)$. The log-likelihood of $y_2, ..., y_T$ conditional on y_1 is obtained as:

$$l(\theta) = -\frac{T}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{t=2}^{T}\epsilon_t^2 = -\frac{T}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{t=2}^{T}(y_t - \alpha y_{t-1} - \phi y_{t-1} G(s_t, \gamma, c))^2$$

To calculate the maximum likelihood estimators, we solve the estimating equations

$$\frac{\partial l(\theta)}{\partial \alpha} = \sum_{t=2}^{T}\sigma^{-2} y_{t-1}\left(y_t - \alpha y_{t-1} - \phi y_{t-1} G(s_t, \gamma, c)\right) = 0$$

$$\frac{\partial l(\theta)}{\partial \phi} = \sum_{t=2}^{T}\sigma^{-2} G(s_t, \gamma, c) y_{t-1}\left(y_t - \alpha y_{t-1} - \phi y_{t-1} G(s_t, \gamma, c)\right) = 0$$

$$\frac{\partial l(\theta)}{\partial \sigma^2} = -\frac{T}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{t=2}^{T}\epsilon_t^2 = 0$$

$$\frac{\partial l(\theta)}{\partial \gamma} = \sum_{t=2}^{T}\sigma^{-2}\left(\phi y_{t-1}\right)\frac{\partial G(s_t, \gamma, c)}{\partial \gamma}\left(y_t - \alpha y_{t-1} - \phi y_{t-1} G(s_t, \gamma, c)\right) = 0 \tag{4}$$

and

$$\frac{\partial l(\theta)}{\partial c} = \sum_{t=2}^{T} \sigma^{-2} (\phi y_{t-1}) \frac{\partial G(s_t, \gamma, c)}{\partial c} (y_t - \alpha y_{t-1} - \phi y_{t-1} G(s_t, \gamma, c)) = 0 \tag{5}$$

The likelihood equations (4 and 5) don't have explicit solution and have to be solved by iterative methods. To formulate modified likelihood equation, we define

$$M_1 (s_t, \gamma, c) = \phi y_{t-1} \frac{\partial G(s_t, \gamma, c)}{\partial \gamma} (y_t - \alpha y_{t-1} - \phi y_{t-1} G(s_t, \gamma, c))$$

and

$$M_2 (s_t, \gamma, c) = \phi y_{t-1} \frac{\partial G(s_t, \gamma, c)}{\partial c} (y_t - \alpha y_{t-1} - \phi y_{t-1} G(s_t, \gamma, c)) .$$

The first two terms of Taylor expansion of $M_1 (s_t, \gamma, c)$ and $M_2 (s_t, \gamma, c)$ around $\gamma = 0$ are

$$M_1 (s_t, \gamma, c) \cong M_1 (\mu_s, \gamma, c) \mid_{\gamma=0} + M_1^{'} (\mu_s, \gamma, c) \mid_{\gamma=0} (\gamma - 0)$$

and

$$M_2 (s_t, \gamma, c) \cong M_2 (\mu_s, \gamma, c) \mid_{\gamma=0} + M_2^{'} (\mu_s, \gamma, c) \mid_{\gamma=0} (\gamma - 0),$$

where $M_1^{'} (\mu_s, \gamma, c)$ and $M_2^{'} (\mu_s, \gamma, c)$ are the derivative of $M_1 (s_t, \gamma, c)$ and $M_2 (s_t, \gamma, c)$ with respect to gamma, respectively. Substituted the Taylor expansion of $M_1 (s_t, \gamma, c)$ and $M_2 (s_t, \gamma, c)$ in (4 and 5) respectively and obtain modified maximum likelihood estimators by solving the estimating equations as:

$$\hat{\alpha} = \left(\sum_{t=2}^{T} y_{t-1}^2 \right)^{-1} \sum_{t=2}^{T} y_{t-1} \left(y_t - \hat{\phi} y_{t-1} G(s_t, \hat{\gamma}, \hat{c}) \right) \tag{6}$$

$$\hat{\phi} = \left(\sum_{t=2}^{T} y_{t-1}^2 G^2 (s_t, \hat{\gamma}, \hat{c}) \right)^{-1} \sum_{t=2}^{T} y_{t-1} G(s_t, \hat{\gamma}, \hat{c}) (y_t - \hat{\alpha} y_{t-1}) \tag{7}$$

and

$$\hat{\sigma}^2 = \frac{1}{T} \sum_{t=2}^{T} \epsilon_t^2. \tag{8}$$

If the transition function be logistic function

$$G(s_t, \gamma, c) = (1 + exp \{-\gamma (s_t - c)\})^{-1} - \frac{1}{2},$$

so

$$\begin{aligned} \frac{\partial l(\theta)}{\partial \gamma} &= \sum_{t=2}^{T} \sigma^{-2} \phi y_{t-1} (s_t - c) e^{-\gamma (s_t - c)} \left(\frac{y_t - \alpha y_{t-1} - \phi y_{t-1} \left[\left(1 + e^{-\gamma (s_t - c)}\right)^{-1} - \frac{1}{2} \right]}{\left(1 + e^{-\gamma (s_t - c)}\right)^2} \right) \\ &\cong \sum_{t=2}^{T} \sigma^{-2} (\phi y_{t-1}) \left(\frac{1}{4} \right) (s_t - c) \left(y_t - \alpha y_{t-1} - (\phi y_{t-1}) \left(-\frac{\gamma}{4} (s_t - c) \right) \right) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial l(\theta)}{\partial c} &= -2\gamma \sum_{t=2}^{T} \sigma^{-2} \phi y_{t-1} e^{-\gamma (s_t - c)} \left(\frac{y_t - \alpha y_{t-1} - \phi y_{t-1} \left[\left(1 + e^{-\gamma (s_t - c)}\right)^{-1} - \frac{1}{2} \right]}{\left(1 + e^{-\gamma (s_t - c)}\right)^2} \right) \\ &\cong -\frac{\gamma}{2} \sum_{t=2}^{T} \sigma^{-2} \phi y_{t-1} \left(y_t - \alpha y_{t-1} - \phi y_{t-1} \left(-\frac{\gamma}{4} (s_t - c) \right) \right) \\ &= 0. \end{aligned}$$

Then the modified maximum likelihood estimator of γ and c are

$$\hat{\gamma} = \frac{4 \sum_{t=2}^{T} \hat{\phi} y_{t-1} (s_t - \hat{c}) (\hat{\alpha} y_{t-1} - y_t)}{\sum_{t=2}^{T} \left(\hat{\phi} y_{t-1} \right)^2 (s_t - \hat{c})^2} \tag{9}$$

and

$$\hat{c} = \frac{4\sum_{t=2}^{T}\hat{\phi}y_{t-1}\left(y_t - \hat{\alpha}y_{t-1} + \frac{\hat{\gamma}\hat{\phi}}{4}s_t y_{t-1}\right)}{\gamma\sum_{t=2}^{T}\left(\hat{\phi}y_{t-1}\right)^2}. \tag{10}$$

If the transition function be exponential function

$$G(s_t,\gamma,c) = 1 - exp\left\{-\gamma(s_t-c)^2\right\},$$

so

$$\begin{aligned}\frac{\partial l(\theta)}{\partial\gamma} &= \sum_{t=2}^{T}\sigma^{-2}\phi y_{t-1}(s_t-c)^2 e^{-\gamma(s_t-c)^2}\left(y_t - \alpha y_{t-1} - \phi y_{t-1}\left(1-e^{-\gamma(s_t-c)^2}\right)\right)\\ &\cong \sum_{t=2}^{T}\sigma^{-2}\phi y_{t-1}(s_t-c)^2(y_t-\alpha y_{t-1}-\phi y_{t-1})(1-\gamma(s_t-c)^2 + \sum_{t=2}^{T}\sigma^{-2}\left(\phi y_{t-1}\right)^2(s_t-c)^2(1-2\gamma(s_t-c)^2)\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial l(\theta)}{\partial c} &= \gamma\sum_{t=2}^{T}\sigma^{-2}\phi y_{t-1}(s_t-c)e^{-\gamma(s_t-c)^2}\left(y_t - \alpha y_{t-1} - \phi y_{t-1}\left(1-e^{-\gamma(s_t-c)^2}\right)\right)\\ &\cong \sum_{t=2}^{T}\sigma^{-2}\phi y_{t-1}(s_t-c)(y_t-\alpha y_{t-1}-\phi y_{t-1})(1-\gamma(s_t-c)^2 + \sum_{t=2}^{T}\sigma^{-2}\left(\phi y_{t-1}\right)^2(s_t-c)(1-2\gamma(s_t-c)^2)\\ &= 0.\end{aligned} \tag{11}$$

Then the modified maximum likelihood estimator of γ is

$$\hat{\gamma} = \frac{\sum_{t=2}^{T}\hat{\phi}y_{t-1}(s_t-\hat{c})^2(y_t-\hat{\alpha}y_{t-1})}{\sum_{t=2}^{T}\hat{\phi}y_{t-1}(s_t-\hat{c})^2(y_t-\hat{\alpha}y_{t-1}+\hat{\phi}y_{t-1})(s_t-\hat{c})^2} \tag{12}$$

and the estimator of c can be calculated by solving the cubic equation of (11). The modified maximum likelihood estimators follow the asymptotic properties of maximum likelihood estimators (such as consistency and asymptotic normal distribution).

Theorem 2.1. *(Asymptotic Distribution of the LR Statistic): Given assumptions of Vuong Vuong (1989): If* $f^{B_*}(.) = g^{\beta_*}(.)$ *then*

$$2LR_T(\hat{B},\hat{\beta}) \xrightarrow{D} M_m(.;\lambda_*),$$

where $M_m(.;\lambda_*)$ *is a weighted sum of chi-squares with some parameters* m *and* λ_**. The parameter* λ_* *is the vector of eigenvalues of*

$$W = \begin{pmatrix} -D_f(B_*)C_f^{-1}(B_*) & -D_{fg}(B_*,\beta_*)C_g^{-1}(\beta_*) \\ D_{gf}(\beta_*,B_*)C_f^{-1}(B_*) & D_g(\beta_*)C_g^{-1}(\beta_*) \end{pmatrix}.$$

If $f^{B_*}(.) \neq g^{\beta_*}(.)$ *then*

$$T^{-\frac{1}{2}}LR_T(\hat{B},\hat{\beta}) - T^{\frac{1}{2}}E_h\left[\log\frac{f^{B_*}(Y)}{g^{\beta_*}(Y)}\right] \xrightarrow{D} N(0,\omega_*^2)$$

where $\omega_*^2 = Var_h\left[\log\frac{f^{B_*}(Y)}{g^{\beta_*}(Y)}\right]$.

3 Simulation Analysis

In this section, we examine by simulation the relative performance of the material presented so far in the paper. In particular, we examine more closely the performance of the modified maximum likelihood method for estimation and also the performance of the model selection test. We consider exponential and logistic smooth transition autoregressive models as true models. The observations are generated from true model $y_t = \alpha y_{t-1} + \phi y_{t-1}G(y_{t-1},\gamma,c) + \epsilon_t$, where $\theta_0 = (\alpha,\phi,\gamma,c) = (0.3,-0.5,2.0,0.4)$ and error term ϵ_t distributed as Normal distribution, $N(0,1)$. Here we assume that the true model is known and only the parameters need to

Table 1: The MMLE of LSTAR and ESTAR models.

	T	$\hat{\alpha}$	$\hat{\phi}$	$\hat{\gamma}$	$\hat{c}$
LSTAR:					
	50	0.2502	-0.6515	2.1262	0.5567
	150	0.2808	-0.6286	1.7296	0.3447
	300	0.2939	-0.5928	1.7976	0.3570
	500	0.3032	-0.5824	1.8043	0.3630
	1000	0.3025	-0.5746	1.9032	0.3776
ESTAR :					
	50	0.1279	-0.8134	1.5940	0.8302
	150	0.3993	-0.6464	1.8073	0.6879
	300	0.3835	-0.5828	1.8096	0.5405
	500	0.3370	-0.5546	1.9163	0.4618
	1000	0.3130	-0.5418	2.0427	0.4245

be estimated. The results for all estimation procedure are given for different sample sizes, of n=50, 150, 300, 500, 1000 and are summarized in Table 1. It shows that, as the sample size increase, the value of estimators become closer to the true parameters.
We perform 10^3 replications. The results for all estimation procedure and their mean square error are given for different sample sizes of n=50, 150, 300, 500, 1000 and are summarized in Table 2 and Table 3. In Table 2 the mean of modified maximum likelihood estimation of parameters are given. In table 3, we present the corresponding mean squared error vis-a-vis the true parameters

$$MSE = \frac{1}{m}\sum_{i=1}^{m}(\theta_0 - \hat{\theta}_i)^2$$

where $m = 10^3$. It shows that, as the sample size increase the mean square error decrease and the value of estimators are closed to the true parameters.

Table 2: The MMLE of LSTAR and ESTAR models with 10^3 replications.

	T	$\hat{\alpha}$	$\hat{\phi}$	$\hat{\gamma}$	$\hat{c}$
LSTAR:					
	50	0.3657	-0.6874	2.9845	0.5421
	150	0.3584	-0.5589	2.7584	0.4875
	300	0.3379	-0.5423	2.4876	0.4662
	500	0.3295	-0.5420	0.2491	0.4550
	1000	0.3198	-0.5264	2.1128	0.4292
ESTAR:					
	50	0.2645	-0.5549	1.3524	0.7313
	150	0.2753	-0.5510	1.5961	0.6110
	300	0.2846	-0.5451	1.6842	0.4990
	500	0.2881	-0.5023	1.8543	0.4227
	1000	0.3010	-0.5022	1.9624	0.4171

The results in Tables 1-3 are for the case of a well-specified model. But what happens when we consider model with a misspecified distribution? Consider therefore that the true model is a logistic smooth transition autoregressive model with $\theta_0 = (\alpha, \phi, \gamma, c) = (0.4, -0.4, 2, 0.4)$. We will ignore the true model and estimate models assuming a LSTAR and ESTAR to examine the impact of misspecification on estimation method. In Table 4, we report the results for the value of the estimated parameters, p-value of the associated Kolmogorov-Smirnov test, KS, and mean-squared error of the estimation of the one-step ahead prediction, $MSE = \frac{1}{T}\sum_{t=1}^{T}(y_t - \hat{y}_t)^2$. We do this to illustrate the potential problems that will arise in a misspecified model when parameters that do not belong to the true model are estimated and then used to make predictions. The results in Table 4 show that, as the sample size increase the mean

Table 3: The MSE of LSTAR and ESTAR models.

	T	$\hat{\alpha}$	$\hat{\phi}$	$\hat{\gamma}$	$\hat{c}$
ESTAR:					
	50	0.6572	0.8754	1.5624	1.8935
	150	0.5986	0.6586	0.8457	0.6342
	300	0.5626	0.6351	0.6930	0.5748
	500	0.4930	0.4932	0.5171	0.4933
	1000	0.3569	0.1008	0.1464	0.3402
LSTAR:					
	50	5.7722	0.3176	2.6934	0.4626
	150	1.7361	0.2735	2.6750	0.3108
	300	0.5634	0.2443	2.4930	0.2665
	500	0.3769	0.2111	1.6930	0.1583
	1000	0.3755	0.1920	1.4235	0.1470

square error decrease and the estimated LSTAR model has the lowest value of the MSE_h.

Table 4: The value of MMLE and P-value of K-S test in misspecification case.

	T	$\hat{\alpha}$	$\hat{\phi}$	$\hat{\gamma}$	$\hat{c}$	KS	MSE_h
LSTAR:							
	50	0.3744	-0.3575	1.6273	0.2583	0.1358	0.1495
	150	0.3757	-0.4207	1.8943	0.3635	0.3524	0.0852
	300	0.4148	-0.4082	1.9127	0.3746	0.6475	0.0163
	500	0.4033	-0.4048	1.9650	0.3863	0.6654	0.0110
	1000	0.4015	-0.4007	2.0166	0.3994	0.8471	0.0076
ESTAR:							
	50	0.5474	-0.3457	2.362	0.8804	0.0000	1.0527
	150	0.5305	-0.5188	1.4739	0.7771	0.0029	0.9532
	300	0.5232	-0.4971	1.4662	0.6456	0.0208	0.1382
	500	0.4224	-0.4468	1.5073	0.6181	0.0586	0.0586
	1000	0.4417	-0.4236	1.5524	0.6144	0.1396	0.1266

For theoretical results of model selection test, let us consider LSTAR model as true model and LSTAR and ESTAR models as competing models. Using obtained data and modified maximum likelihood method, the value of Vuong's statistic for paired of competing models are computed and compared with $z_{0.975} = 1.96$. The obtained value of Vuong's test statistic are given in Table 5. We observe that LSTAR model is optimum model. The Kolmogorov-Smirnov test presented in Table 4 confirms these results. For more illustration see Figure 1-2.

Also the relative frequency of model selection test for each of rejection-acceptance regions is

Table 5: The values of Vuong's test.

T	$H_f : LSTAR$ $H_g : ESTAR$	Result
50	3.8422	LSTAR is better than ESTAR
150	4.5245	LSTAR is better than ESTAR
300	5.5764	LSTAR is better than ESTAR
500	7.2386	LSTAR is better than ESTAR
1000	9.9750	LSTAR is better than ESTAR

computed and the results are summarized in Table 6. For example, if n=300, when we test $H_f : LSTAR$ against $H_g : ESTAR$, the value of the relative frequency is 0.995, see column 2. It shows that Vuong's test select estimated LSTAR model as optimum model.

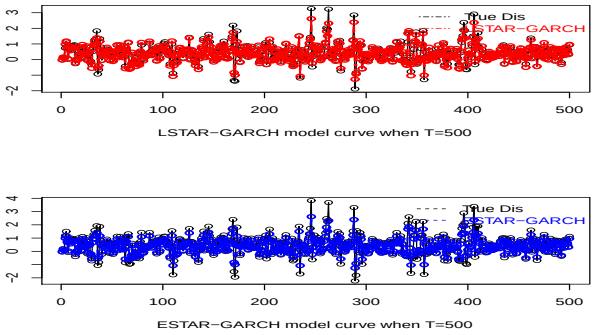


Figure 1: The estimated LSTAR and ESTAR models curve when T=500.

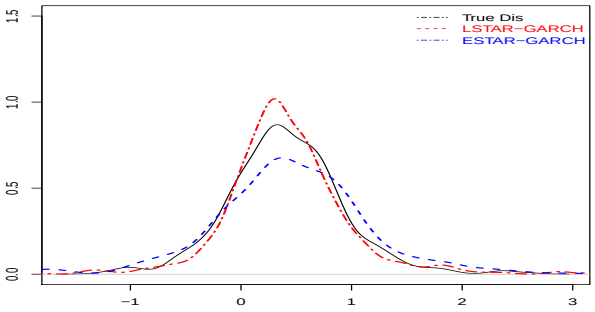


Figure 2: The competing model curve when T=500.

Table 6: The relative frequency of Vuong results.

T	f better	g better	f, g equi
50	0.823	0.097	0.080
150	0.945	0.053	0.002
300	0.995	0.005	0.000
500	0.999	0.001	0.000
1000	1.000	0.000	0.000

4 Conclusion

In this paper, we make a number of contributions to the literature that relates to nonlinear time series models. First, we proposed the modified maximum likelihood estimation for smooth transition autoregressive model. This is important, because explicit solutions from the likelihood equation cannot be obtained. The modified maximum likelihood estimators are consistent and asymptotically normally distributed. Also, the Vuong's test is proposed as model selection test based on the modified maximum likelihood estimator. Second, we examined the properties of the estimators and model selection statistic using simulations which validate our theoretical results. In summary, the theoretical derivations and the simulations support the use of the proposed model selection test as a useful tool in a practitioner's toolbox for empirical analysis.

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