پانزدهمین سمینار احتمال
و فرآیندهای تصادفی
۸ و ۹ شهریور ۱۴۰۴
دانشگاه کردستان

15th Seminar On Probability and Stochastic Processes
August 30-31, 2025
University of Kurdistan

انجمن آمار ایران

Complete convergence for randomly weighted sums of WOD sequences

R.Javanmard*,A.Nezakati

Faculty of Mathematical Sciences, Shahrood University of Technology, Shahrood, Iran

Abstract

In this paper, we will study the complete convergence for double-indexed randomly weighted sums of WOD random variables. Several sufficient conditions to prove the complete convergence for randomly weighted sums of WOD random variables are presented.

Keywords: WOD random variables;randomly weighted sums;complete convergence;stochastically dominated.

Mathematics Subject Classification (2020): 60F15,62G05.

1 Introduction

In P. L. Hsu, H. Robbins (1947) if $\{X, X_n, n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d., for short) random variables and set $S_n = \sum_{i=1}^{n} X_i$ such that $EX^2 < \infty$ and $EX = \mu$, then for all $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(\left|n^{-1}S_n - \mu\right| > \epsilon\right) < \infty.$$

The aforementioned result is the well-known **complete convergence**, which implies that $S_n/n \to 0$ almost surely (a.s., for short) by the Borel-Cantelli lemma.
In Baum, L. E., and M. Katz (1965) has been proven For a sequence $\{X, X_n, n \geq 1\}$ of i.i.d. random variables with zero mean and for all $\epsilon > 0$,

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\left|\sum_{i=1}^{n} X_i\right| > \epsilon n^{1/p}\right) < \infty$$

if, and only if, $E|X|^{rp} < \infty$, where $r > 1$ and $1 \leq p < 2$. Since its first appearance in 1947, many authors have made enormous achievements in this field. In many stochastic models, the assumption that random variables are independent is not plausible. So it is interesting to extend the concept of independence to dependence cases. One of these dependence structures is widely orthant dependent (WOD),which has important applications in time series and fnancial mathematics. Many scholars studied the convergence properties for weighted sums of WOD random variables, and obtained many interesting results. In this work, we will further study the strong convergence properties for randomly weighted sums of WOD random variables.

*email:Rezvan.Javanmard@shahroodut.ac.ir

Now let us recall the concept of WOD random variables.which was raised by K. Y. Wang, Y. B. Wang, Q. W. Gao (2013) as follows.

Definition 1.1. *A sequence $\{X_n, n \geq 1\}$ of random variables is said to be widely upper orthant dependent (WUOD) if there exists a finite sequence $\{g_U(n), n \geq 1\}$ of positive numbers such that for each $n \geq 1$ and for all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,*

$$P(X_1 > x_1, X_2 > x_2, \cdots, X_n > x_n) \leq g_U(n) \prod_{i=1}^{n} P(X_i > x_i).$$

A sequence $\{X_n, n \geq 1\}$ is widely lower orthant dependent (WLOD) if there exists a finite sequence $\{g_L(n), n \geq 1\}$ of positive numbers such that for each $n \geq 1$ and for all $x_i \in (-\infty, \infty)$, $1 \leq i \leq n$,

$$P(X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n) \leq g_L(n) \prod_{i=1}^{n} P(X_i \leq x_i).$$

If $\{X_n, n \geq 1\}$ is both WUOD and WLOD, then it is said to be WOD with dominating coefficients $g(n) = \max\{g_U(n), g_L(n)\}$.
The defnitions of stochastic domination and slowly varying function below will be used throughout the paper.

Definition 1.2. *A triangular array $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ of random variables is said to be **stochastically dominated** by a random variable X, if there exists a positive constant C such that*

$$P(|X_{nk}| > x) \leq CP(|X| > x), \quad \text{for any } x \geq 0,\ 1 \leq k \leq n,\ \text{and } n \geq 1.$$

Definition 1.3. *Let $l(x)$ be a real-valued function, positive and measurable on $(0, \infty)$. It is **slowly varying at infinity** if for any $t > 0$,*

$$\lim_{x \to \infty} \frac{l(tx)}{l(x)} = 1.$$

Baum, L. E., and M. Katz (1965) studied the complete convergence for a sequence of independent identical distributed random variables. Then the theorems for independent random variables were extended to the cases of dependent random variables. Recently, Wang, X., C. Xu, T.-C. Hu, A. Volodin, and S. Hu (2014) and Wang, X. J., Y. Wu, and A. Rosalsky (2017) have studied the complete convergence for weighted sums of WOD random variables. Dawei Lu Jialu Wang (2021) investigate the complete convergence and the complete moment convergence for maximal randomly weighted sums of a triangular array of row-wise WOD random variables. Now many people pay attention to the randomly weighted sums of random variables. Yi Wu and Duoduo Zhao (2024) further investigate the single logarithm rule in the full sense of convergence for randomly weighted sums of WOD, where the weights are also assumed to be WOD.
In this paper, we will study the complete convergence for double-indexed randomly weighted sums of WOD where the weights are independent and nonnegatine random variables . Several sufficient conditions to prove the complete convergence for randomly weighted sums of WOD random variables are presented.The result improves and generalizes some corresponding ones in the literature and the conditions considered in our paper are better and easier than Yi Wu and Duoduo Zhao (2024) .

2 Preliminary lemmas

*In this section, we provide some lemmas to prove our main results. The frst one is a basic property for WOD random variables, which can be found in Dawei Lu and Jialu Wang. To prove the main results, we list three assumptions (**H1–H3**) and present some important lemmas in the following. To prove the main results, we list three assumptions (H1–H3) and present some important lemmas in the following.*

(H1) *Let* $\{X_n, n \geq 1\}$ *be a sequence of WOD random variables with dominating coefficients* $g_u(m)$, $g_l(m)$, $m \geq 1$. *Denote* $g(m) = \max\{g_u(m), g_l(m)\}$.

(H2) *Let* $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ *be a triangular array of row-wise WOD random variables with dominating coefficients* $g_u^{(n)}(m)$, $g_l^{(n)}(m)$, $1 \leq m \leq n$, $n \geq 1$ *in each row. Denote* $g^{(n)}(m) = \max\{g_u^{(n)}(m), g_l^{(n)}(m)\}$.

(H3) *Let* $\{A_{ni}, 1 \leq i \leq n, n \geq 1\}$ *be a triangular array of row-wise independent random variables, which are independent of* $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$.

Here a triangular array of random variables $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ *are called row-wise WOD random variables if* $\{X_{ni}, 1 \leq i \leq n\}$ *is a sequence of WOD random variables for every* $n \geq 1$. *Then we show a basic property of WOD random variables in the following lemma. See Wang, X., C. Xu, T.-C. Hu, A. Volodin, and S. Hu (2014) for more details.*

Lemma 2.1. *Let* $\{X_n, n \geq 1\}$ *be a sequence of WOD random variables with dominating coefficients* $g(m)$, $m \geq 1$*:*

(i) $\{f_n(X_n), n \geq 1\}$ *remains a sequence of WOD random variables with the same dominating coefficients* $g(m), m \geq 1$, *provided* $\{f_n(\cdot), n \geq 1\}$ *are all nondecreasing (or all nonincreasing) functions.*

(ii) *For each* $m \geq 1$ *and all* $s \in \mathbb{R}$*:*

$$E \exp\left\{s \sum_{i=1}^{m} X_i\right\} \leq g(m) \prod_{i=1}^{m} E \exp\{sX_i\}$$

According to Lemma 2.1 and the definition of WOD random variables, we can obtain the following corollaries easily.

Corollary 2.1. *Let* $\{X_n, n \geq 1\}$ *be a sequence of WOD random variables and* $\{Y_n, n \geq 1\}$ *be a sequence of nonnegative and independent random variables. If the two sequences are independent with each other, then* $\{X_n Y_n, n \geq 1\}$ *is still a sequence of WOD random variables.*

In the following, we show some pivotal inequalities for WOD random variables. The most significant inequalities are Marcinkiewicz-Zygmund type maximal inequality and Rosenthal type maximal inequality given in the lemma below.See Ding, Y., Y. Wu, S. Ma, X. Tao, and X. Wang (2017) for more details.

Lemma 2.2. *Let* $p \geq 1$ *and* $\{X_n, n \geq 1\}$ *be a sequence of WOD random variables with* $\mathbb{E}X_n = 0$ *and* $\mathbb{E}|X_n|^p < \infty$ *for each* $n \geq 1$. *Then, for* $m \geq 1$, *there exist positive constants* $C_1(p)$ *and* $C_2(p)$ *depending only on* p *such that:*

$$\mathbb{E}\left(\max_{1 \leq k \leq m} \left|\sum_{i=1}^{k} X_i\right|^p\right) \leq [C_1(p) + C_2(p)g(m)] (\log m)^p \sum_{i=1}^{m} \mathbb{E}|X_i|^p$$

for $1 \leq p \leq 2$, *and*

$$\mathbb{E}\left(\max_{1 \leq k \leq m} \left|\sum_{i=1}^{k} X_i\right|^p\right) \leq C_1(p)(\log m)^p \sum_{i=1}^{m} \mathbb{E}|X_i|^p + C_2(p)g(m)(\log m)^p \left(\sum_{i=1}^{m} \mathbb{E}|X_i|^2\right)^{p/2}$$

for $p > 2$.

The following lemma can be found in Yi Wu and Duoduo Zhao (2024).

Lemma 2.3. *Let* $\{Y_i, 1 \leq i \leq n\}$ *and* $\{Z_i, 1 \leq i \leq n\}$ *be two sequences of random variables. Then for any* $t > q > 0$, $\varepsilon > 0$, *and* $a > 0$*:*

$$\mathbb{E}\left[\left(\max_{1 \leq m \leq n} \left|\sum_{i=1}^{m} (Y_i + Z_i)\right| - \varepsilon a\right)_+^q\right] \leq C_q \left(\varepsilon^{-t} + \frac{t-q}{q}\right) a^{q-t} \mathbb{E}\left[\max_{1 \leq m \leq n} \left|\sum_{i=1}^{m} Y_i\right|^t\right]$$

$$+C_q\mathbb{E}\left[\max_{1\le m\le n}\left|\sum_{i=1}^{m}Z_i\right|^q\right],$$

where $C_q = 1$ *if* $0 < q \le 1$, *or* $C_q = 2^{q-1}$ *if* $q > 1$

The following lemma is an important property for stochastic domination, which can be found in Wu Y, Wang XJ, Hu SH (2017) or Shen AT, Zhang Y, Volodin A (2015).

Lemma 2.4. *Let* $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ *be a triangular array of random variables stochastically dominated by a random variable* X. *For any* $a > 0$ *and* $b > 0$:

$$\mathbb{E}|X_{nk}|^a\mathbf{I}(|X_{nk}| \le b) \le C_1\left[\mathbb{E}|X|^a\mathbf{I}(|X| \le b) + b^a\mathbb{P}(|X| > b)\right]$$

$$\mathbb{E}|X_{nk}|^a\mathbf{I}(|X_{nk}| > b) \le C_2\mathbb{E}|X|^a\mathbf{I}(|X| > b)$$

where C_1 *and* C_2 *are positive constants. Consequently:*

$$\mathbb{E}|X_{nk}|^a \le C\mathbb{E}|X|^a$$

for some positive constant C.

The next one is important to prove the main result from Chao Lu, Xiaoqin Li, Rui Wang Xuejun Wang (2020)

Lemma 2.5. *Let* $r \ge 1$, $0 < p \le 2$, $\alpha > 0$, $\beta > 0$ *with* $1/\alpha + 1/\beta = 1/p$. *Let* $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ *be an array of random variables stochastically dominated by* X, *and* $\{A_{nk}\}$ *a triangular array of random variables satisfying* $\sum_{k=1}^{n}|A_{nk}|^\alpha \le Cn$ *a.s., independent of* $\{X_{nk}\}$. *Then:*

$$\sum_{n=1}^{\infty}n^{r-2}l(n)\sum_{k=1}^{n}\mathbb{P}\left(|A_{nk}X_{nk}| > n^{1/p}\right) \le \begin{cases} C\mathbb{E}|X|^{(r-1)\beta}l(|X|^\beta), & \alpha < rp \\ C\mathbb{E}|X|^{(r-1)\beta}\log(1+|X|)l(|X|^\beta), & \alpha = rp \\ C\mathbb{E}|X|^{rp}l(|X|^p), & \alpha > rp \end{cases}$$

We give One remark on the condition $\sum_{k=1}^{n}|A_{nk}|^\alpha \le Cn$ *a.s.*

Remark 2.2. *By* $\sum_{k=1}^{n}|A_{nk}|^\alpha \le Cn$ *a.s., we have*

$$\sum_{k=1}^{n}|A_{nk}|^s \le \begin{cases} Cn & a.s.,\ for\ any\ 0 < s \le \alpha, \\ Cn^{s/\alpha} & a.s.,\ for\ any\ s > \alpha, \end{cases}$$

and thus,

$$\sum_{k=1}^{n}E|A_{nk}|^s \le \begin{cases} Cn & for\ any\ 0 < s \le \alpha, \\ Cn^{s/\alpha} & for\ any\ s > \alpha. \end{cases}$$

Lemma 2.6. *Let* $r \ge 1$, $0 < p \le 2$, $\alpha > 0$, $\beta > 0$ *with* $1/\alpha + 1/\beta = 1/p$. *Let* $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ *be a triangular array of random variables stochastically dominated by* X, *and* $\{A_{nk}\}$ *a triangular array of random variables satisfying* $\sum_{k=1}^{n}|A_{nk}|^\alpha \le Cn$ *a.s., independent of* $\{X_{nk}\}$. *Then for any* $s > \max\{\alpha, (r-1)\beta\}$:

$$\sum_{n=1}^{\infty}n^{r-2-s/p}l(n)\sum_{k=1}^{n}\mathbb{E}\left[|A_{nk}X_{nk}|^s\mathbf{I}\left(|A_{nk}X_{nk}| \le n^{1/p}\right)\right]$$
$$\le \begin{cases} C\mathbb{E}|X|^{(r-1)\beta}l(|X|^\beta), & \alpha < rp \\ C\mathbb{E}|X|^{(r-1)\beta}\log(1+|X|)l(|X|^\beta), & \alpha = rp \\ C\mathbb{E}|X|^{rp}l(|X|^p), & \alpha > rp \end{cases}$$

The proof is in Chao Lu, Xiaoqin Li, Rui Wang Xuejun Wang (2020) The next one is the randomly weighted version of Lemma 2.4 in Chen PY, Sung SH (2018).

Lemma 2.7. *Let $q > 0$, $r \geq 1$, $0 < p \leq 2$, $\alpha > 0$, $\beta > 0$ with $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{p}$. Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be a triangular array of random variables stochastically dominated by X, and $\{A_{nk}\}$ a triangular array of random variables with $\sum_{k=1}^{n} |A_{nk}|^{\alpha} \leq Cn$ a.s., independent of $\{X_{nk}\}$. Then:*

1. *If $\alpha < rp$:*

$$\sum_{n=1}^{\infty} n^{r-2-q/p} l(n) \sum_{k=1}^{n} \mathbb{E}\left[|A_{nk}X_{nk}|^{q} I\left(|A_{nk}X_{nk}| > n^{1/p}\right)\right]$$
$$\leq \begin{cases} C\,\mathbb{E}\,|X|^{(r-1)\beta} l(|X|^{\beta}), & q < (r-1)\beta \\ C\,\mathbb{E}\,|X|^{(r-1)\beta} \log(1+|X|) l(|X|^{\beta}), & q = (r-1)\beta \\ C\,\mathbb{E}\,|X|^{q} l(|X|^{\beta}), & q > (r-1)\beta \end{cases}$$

2. *If $\alpha = rp$:*

$$\sum_{n=1}^{\infty} n^{r-2-q/p} l(n) \sum_{k=1}^{n} \mathbb{E}\left[|A_{nk}X_{nk}|^{q} I\left(|A_{nk}X_{nk}| > n^{1/p}\right)\right]$$
$$\leq \begin{cases} C\,\mathbb{E}\,|X|^{(r-1)\beta} \log(1+|X|) l(|X|^{\beta}), & q \leq \alpha = rp \\ C\,\mathbb{E}\,|X|^{q} l(|X|^{\beta}), & q > \alpha = rp \end{cases}$$

3. *If $\alpha > rp$:*

$$\sum_{n=1}^{\infty} n^{r-2-q/p} l(n) \sum_{k=1}^{n} \mathbb{E}\left[|A_{nk}X_{nk}|^{q} I\left(|A_{nk}X_{nk}| > n^{1/p}\right)\right]$$
$$\leq \begin{cases} C\,\mathbb{E}\,|X|^{rp} l(|X|^{p}), & q < rp \\ C\,\mathbb{E}\,|X|^{rp} \log(1+|X|) l(|X|^{p}), & q = rp \\ C\,\mathbb{E}\,|X|^{q} l(|X|^{p}), & q > rp \end{cases}$$

Lemma 2.8. *Let $1 \leq p \leq 2$, $\alpha > 0$, $\beta > 0$ with $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{p}$. Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be a triangular array of random variables stochastically dominated by X, and $\{A_{nk}\}$ a triangular array of random variables with $\sum_{k=1}^{n} |A_{nk}|^{\alpha} \leq Cn$ a.s., independent of $\{X_{nk}\}$. If $\mathbb{E}|X|^{p} < \infty$, then:*

$$n^{-1/p} \sum_{k=1}^{n} \mathbb{E}\left[|A_{nk}X_{nk}| I\left(|A_{nk}X_{nk}| > n^{1/p}\right)\right] \to 0 \quad \text{as } n \to \infty.$$

Additionally, if $\mathbb{E}X_{nk} = 0$ for all n, k, then:

$$n^{-1/p} \max_{1 \leq m \leq n} \left| \sum_{k=1}^{m} \mathbb{E}\left[A_{nk}X_{nk} I\left(|A_{nk}X_{nk}| \leq n^{1/p}\right)\right] \right| \to 0 \quad \text{as } n \to \infty.$$

The proof is in Chao Lu, Xiaoqin Li, Rui Wang Xuejun Wang (2020).

3 Main result and its proof

With preliminaries in Section 2 accounted for, we can give our main results and their proofs.

Theorem 3.1. *Let $r \geq 1$, $1 \leq p < 2$, $\alpha > 0$, $\beta > 0$ with $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{1}{p}$. Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be a triangular array of rowwise WOD random variables with dominating coefficient $g(n) = O(n^{\tau})$ for some $\tau \geq 0$, stochastically dominated by a random variable X with $\mathbb{E}\,X_{nk} = 0$ and $\mathbb{E}\,|X|^{p} < \infty$. Let $\{A_{nk}, 1 \leq k \leq n, n \geq 1\}$ be a triangular array of rowwise independent and nonnegative random variables satisfying*

$$\sum_{k=1}^{n} |A_{nk}|^{\alpha} \leq Cn \quad \text{a.s.}$$

for some constant $C > 0$, and independent of $\{X_{nk}, 1 \le k \le n, n \ge 1\}$. If

$$\begin{aligned}
&\mathbb{E}\left[|X|^{(r-1)\beta+2\tau}\, l(|X|^{\beta})\right] < \infty, && \text{if } \alpha < rp,\\
&\mathbb{E}\left[|X|^{(r-1)\beta+2\tau}\, \log(1+|X|)\, l(|X|^{\beta})\right] < \infty, && \text{if } \alpha = rp,\\
&\mathbb{E}\left[|X|^{rp+2\tau}\, l(|X|^{p})\right] < \infty, && \text{if } \alpha > rp,
\end{aligned}$$

then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{r-2} l(n)\, \mathbb{P}\left(\max_{1\le m\le n}\left|\sum_{k=1}^{m} A_{nk}X_{nk}\right| > \varepsilon n^{1/p}\right) < \infty. \tag{1}$$

where l(x) is slowly varying at infnity.

Proof. For $1 \le k \le n$ and $n \ge 1$, denote

$$Y_{nk} = -n^{1/p}\mathbf{I}\left(A_{nk}X_{nk} < -n^{1/p}\right) + A_{nk}X_{nk}\mathbf{I}\left(|A_{nk}X_{nk}| \le n^{1/p}\right) + n^{1/p}\mathbf{I}\left(A_{nk}X_{nk} > n^{1/p}\right),$$

$$\begin{aligned}
Z_{nk} &= A_{nk}X_{nk} - Y_{nk}\\
&= \left(A_{nk}X_{nk} + n^{1/p}\right)\mathbf{I}\left(A_{nk}X_{nk} < -n^{1/p}\right) \quad + \left(A_{nk}X_{nk} - n^{1/p}\right)\mathbf{I}\left(A_{nk}X_{nk} > n^{1/p}\right).
\end{aligned}$$

We obtain that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} l(n)\mathbb{P}\left(\max_{1\le m\le n}\left|\sum_{k=1}^{m} A_{nk}X_{nk}\right| > \varepsilon n^{1/p}\right)\\
&\quad = C\sum_{n=1}^{\infty} n^{r-2} l(n)\mathbb{P}\left(\max_{1\le m\le n}\left|\sum_{k=1}^{m}(Y_{nk} - \mathbb{E}[Y_{nk}] + Z_{nk} - \mathbb{E}[Z_{nk}])\right| > \varepsilon n^{1/p}\right)\\
&\quad \le C\sum_{n=1}^{\infty} n^{r-2} l(n)\mathbb{P}\left(\max_{1\le m\le n}\left|\sum_{k=1}^{m}(Y_{nk} - \mathbb{E}[Y_{nk}])\right| > \frac{1}{2}\varepsilon n^{1/p}\right)\\
&\qquad + C\sum_{n=1}^{\infty} n^{r-2} l(n)\mathbb{P}\left(\max_{1\le m\le n}\left|\sum_{k=1}^{m}(Z_{nk} - \mathbb{E}[Z_{nk}])\right| > \frac{1}{2}\varepsilon n^{1/p}\right)\\
&\quad \triangleq I_1 + I_2.
\end{aligned}$$

At first, we consider the case of $r > 1$. For I_2, noting that when $\alpha \le rp$, we have by $1/\alpha + 1/\beta = 1/p$ that $\alpha = p\beta/(\beta - p) \le rp$, and thus, $(r-1)\beta \ge rp > 1$. Similarly, we also get that $rp > \max\{(r-1)\beta, 1\}$ when $\alpha > rp$. Thus, we know that there exist some real number q such that $1 \le q < (r-1)\beta$ if $\alpha \le rp$, and $1 \le q < rp$ if $\alpha > rp$. Taking $q = 1$ in lemma 2.7, we have by Markov's inequality that

$$\begin{aligned}
I_2 &= \sum_{n=1}^{\infty} n^{r-2} l(n)\mathbb{P}\left(\max_{1\le m\le n}\left|\sum_{k=1}^{m}(Z_{nk} - \mathbb{E}[Z_{nk}])\right| > \frac{1}{2}\varepsilon n^{1/p}\right)\\
&\le C\sum_{n=1}^{\infty} n^{r-2-1/p} l(n)\sum_{k=1}^{n}\mathbb{E}[|A_{nk}X_{nk}|]\mathbf{I}\left(|A_{nk}X_{nk}| > n^{1/p}\right) < \infty.
\end{aligned}$$

Note that

$$\mathbb{E}\left[\max_{1\le m\le n}\left|\sum_{k=1}^{m}(Z_{nk} - \mathbb{E}[Z_{nk}])\right|\right] \le \mathbb{E}\left[\left|\sum_{k=1}^{n}(Z_{nk} - \mathbb{E}[Z_{nk}])\right|\right]$$

$$\mathbb{E}\left[\sum_{k=1}^{n}|Z_{nk} - \mathbb{E}[Z_{nk}]|\right] = \sum_{k=1}^{n}\mathbb{E}\left[|Z_{nk} - \mathbb{E}[Z_{nk}]|\right]$$

$$\mathbb{E}\left[|Z_{nk} - \mathbb{E}[Z_{nk}]|\right] \le 2\,\mathbb{E}[|Z_{nk}|] \quad \text{and} \quad \mathbb{E}[|Z_{nk}|] \le \mathbb{E}\left[|A_{nk}X_{nk}| \cdot I_{\{|A_{nk}X_{nk}| > n^{1/p}\}}\right]$$

For I_1, without loss of generality, we assume that $\max_{1\le k\le n}|A_{nk}| \le n^{1/\alpha}$ holds almost surely. Noting that $r > 1$, we set $p < s < \min\left\{\frac{p(r+2\tau)}{\tau+1}, 2, \alpha\right\}$ if $\alpha \le rp$, and $p < s <$

$\min\left\{\frac{p(r+2\tau)}{\tau+1}, 2, rp\right\}$ if $\alpha > rp$. Taking $q > \max\left\{\alpha, (r-1)\beta, , 2\tau+2, p(r-1+\tau), \frac{2p(r-1+\tau)}{s-p}\right\}$, we obtain by Markov's inequality and Lemma 2.2 that:

$$\begin{aligned}
I_1 &= \sum_{n=1}^{\infty} n^{r-2} l(n) \mathbb{P}\left(\max_{1\le m\le n}\left|\sum_{k=1}^{m}(Y_{nk}-\mathbb{E}\,Y_{nk})\right| > \tfrac{1}{2}\varepsilon n^{1/p}\right)\\
&\le C\sum_{n=1}^{\infty} n^{r-2-q/p} l(n)\,\mathbb{E}\left(\max_{1\le m\le n}\left|\sum_{k=1}^{m}(Y_{nk}-\mathbb{E}\,Y_{nk})\right|\right)^{q}\\
&\le C_1\sum_{n=1}^{\infty} n^{r-2-q/p+\tau} l^*(n)\left(\sum_{k=1}^{n}\mathbb{E}\,|Y_{nk}-\mathbb{E}\,Y_{nk}|^2\right)^{q/2}\\
&\quad + C_2\sum_{n=1}^{\infty} n^{r-2-q/p} l(n)(\log n)^q \sum_{k=1}^{n}\mathbb{E}\,|Y_{nk}-\mathbb{E}\,Y_{nk}|^q\\
&\triangleq I_3 + I_4.
\end{aligned}$$

where $l^*(n) = l(n)(\log n)^q$ is a slowly varying function at infinity.
For I_4, *by Lemmas 2.5 and 2.6, we obtain*

$$\sum_{n=1}^{\infty} n^{r-2-q/p} l(n)\sum_{k=1}^{n}\mathbb{E}|Y_{nk}-\mathbb{E}Y_{nk}|^q \le C\sum_{n=1}^{\infty} n^{r-2-q/p} l(n)\times$$
$$\sum_{k=1}^{n}\left[\mathbb{E}|A_{nk}X_{nk}|^q I(|A_{nk}X_{nk}|\le n^{1/p}) + n^{q/p}P(|A_{nk}X_{nk}|>n^{1/p})\right] < \infty.$$

For I_3, we have by C_r-inequality and Markov's that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2-q/p+\tau} l^*(n)\left(\sum_{k=1}^{n}\mathbb{E}(Y_{nk}-\mathbb{E}Y_{nk})^2\right)^{q/2}\\
&\le \sum_{n=1}^{\infty} n^{r-2-q/p+\tau} l^*(n)\\
&\quad\times\left(\sum_{k=1}^{n}\left[\mathbb{E}|A_{nk}X_{nk}|^2 I(|A_{nk}X_{nk}|\le n^{1/p}) + n^{2/p}P(|A_{nk}X_{nk}|>n^{1/p})\right]\right)^{q/2}\\
&\le C\sum_{n=1}^{\infty} n^{r-2-q/p+\tau} l^*(n)\left(\sum_{k=1}^{n}\mathbb{E}|A_{nk}X_{nk}|^2 I(|A_{nk}X_{nk}|\le n^{1/p})\right)^{q/2}\\
&\quad + C\sum_{n=1}^{\infty} n^{r-2-q/p+\tau} l^*(n)\left(\sum_{k=1}^{n} n^{2/p}P(|A_{nk}X_{nk}|>n^{1/p})\right)^{q/2}\\
&\le C\sum_{n=1}^{\infty} n^{r-2-q/p+\tau} l^*(n)\left(n^{(2-s)/p}\sum_{k=1}^{n}\mathbb{E}|A_{nk}X_{nk}|^s I(|A_{nk}X_{nk}|\le n^{1/p})\right)^{q/2}\\
&\quad + C\sum_{n=1}^{\infty} n^{r-2-q/p+\tau} l^*(n)\left(n^{(2-s)/p}\sum_{k=1}^{n}\mathbb{E}|A_{nk}X_{nk}|^s\right)^{q/2}\\
&\le C\sum_{n=1}^{\infty} n^{r-2-q/p+\tau} l^*(n)\left(n^{(2-s)/p}\sum_{k=1}^{n}\mathbb{E}|A_{nk}|^s\mathbb{E}|X|^s\right)^{q/2} < \infty
\end{aligned}$$

Note that

$$\begin{aligned}
&E\left[|A_{nk}X_{nk}|^s I\left(|A_{nk}X_{nk}|\le n^{1/p}\right)\right] \le E\left[|A_{nk}X_{nk}|^s\right]\\
&\sum_{n=1}^{\infty} n^{r-2-q/p+\tau} l^*(n)\left(n^{(2-s)/p}\sum_{k=1}^{n}E|A_{nk}X_{nk}|^s\right)^{q/2}\\
&\le C\sum_{n=1}^{\infty} n^{r-2-q/p+\tau} l^*(n)\left(n^{(2-s)/p}E|X|^s\sum_{k=1}^{n}|A_{nk}|^s\right)^{q/2}
\end{aligned}$$

and with use from Remark 2.2 and Since

$$n^{r-2-q/p+\tau}\cdot\left(n^{(2-s)/p}\cdot n\right)^{q/2}=n^{r-2+\tau+(q/2)(1-s/p)}$$

and

$$r-2+\tau+\frac{q}{2}\left(1-\frac{s}{p}\right)<-1$$

if

$$q>\frac{2p(r-1+\tau)}{s-p}$$

then

$$\sum_{n=1}^{\infty}n^{r-2+\tau+\frac{q}{2}(1-s/p)}\cdot l^*(n)<\infty$$

Next, we will consider the case of $r=1$. For any $1\le k\le n$ and $n\ge 1$, set

$$\left\{\max_{1\le m\le n}\left|\sum_{k=1}^{m}A_{nk}X_{nk}\right|>\varepsilon n^{1/p}\right\}\subseteq\bigcup_{k=1}^{n}\left\{|A_{nk}X_{nk}|>n^{1/p}\right\}\cup\left\{\max_{1\le m\le n}\left|\sum_{k=1}^{m}T_{nk}\right|>\varepsilon n^{1/p}\right\}.$$

Thus, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}n^{r-2}l(n)\mathbb{P}\left(\max_{1\le m\le n}\left|\sum_{k=1}^{m}A_{nk}X_{nk}\right|>\varepsilon n^{1/p}\right)\\&\le\sum_{n=1}^{\infty}n^{r-2}l(n)\mathbb{P}\left(\bigcup_{k=1}^{n}\left\{|A_{nk}X_{nk}|>n^{1/p}\right\}\right)+\sum_{n=1}^{\infty}n^{r-2}l(n)\mathbb{P}\left(\max_{1\le m\le n}\left|\sum_{k=1}^{m}T_{nk}\right|>\varepsilon n^{1/p}\right)\\&\le\sum_{n=1}^{\infty}n^{r-2}l(n)\sum_{k=1}^{n}\mathbb{P}\left(|A_{nk}X_{nk}|>n^{1/p}\right)+\sum_{n=1}^{\infty}n^{r-2}l(n)\mathbb{P}\left(\max_{1\le m\le n}\left|\sum_{k=1}^{m}T_{nk}\right|>\varepsilon n^{1/p}\right).\end{aligned}$$

By Lemmas 2.5 and 2.6, to prove 1, it suffices to prove that

$$\sum_{n=1}^{\infty}n^{r-2}l(n)P\left(\max_{1\le m\le n}\left|\sum_{k=1}^{m}(T_{nk}-ET_{nk})\right|>\varepsilon n^{1/p}\right)<\infty.$$

In fact, by $r=1$, we have $\alpha>p=rp$. If $\sum_{k=1}^{n}|A_{nk}|^{\alpha}\le Cn$ a.s. for some $\alpha>0$, then we can get that $\sum_{k=1}^{n}|A_{nk}|^{\alpha}\le Cn$ a.s. for any $0<\alpha\le\alpha$ by Remark 2.2 Thus we may assume that $p<\alpha<2$. By Markov's inequality and Lemmas 2.2 and 2.6, we have

$$\begin{aligned}&\sum_{n=1}^{\infty}n^{r-2}l(n)P\left(\max_{1\le m\le n}\left|\sum_{k=1}^{m}(T_{nk}-\mathbb{E}\,T_{nk})\right|>\varepsilon n^{1/p}\right)\\&\le C\sum_{n=1}^{\infty}n^{r-2}\cdot n^{-2/p}l^*(n)\sum_{k=1}^{n}\mathbb{E}\left[|A_{nk}X_{nk}|^2\mathbf{I}\left(|A_{nk}X_{nk}|\le n^{1/p}\right)\right]<\infty.\end{aligned}$$

□

References

K. Y. Wang, Y. B. Wang, Q. W. Gao (2013), Uniform asymptotics for the fnite-time ruin probability of a new dependent risk model with a constant interest rate, Methodol. Comput. Appl. Probab., 15 (2013), 109124. DOI:10.1007/s11009-011-9226-y.

Wang, X. J., Y. Wu, and A. Rosalsky (2017),Complete convergence for arrays of rowwise widely orthant dependent random variables and its applications. Stochastics - An International Journal of Probability and Stochastic Processes 89 (8):122852. DOI:10.1080/17442508.2017.1297813.

Wu Y, Wang XJ, Hu SH(2017)Complete moment convergence for weighted sums of weakly dependent random variables and its application in nonparametric regression model.Statist Probab Lett. 2017;127:5666.

Wang, X., C. Xu, T.-C. Hu, A. Volodin, and S. Hu (2014),On complete convergence for widely orthant-dependent random variables and its applications in nonparametric regression models.Test 23 (3):60729. DOI:10.1007/s11749-014-0365-7.

Yi Wu and Duoduo Zhao (2024), Law of the single logarithm for randomly weighted sums of dependent sequences and an application, AIMS Mathematics, 9(4): 1014110156.

Dawei Lu Jialu Wang (2021),Complete convergence and complete moment convergence for maximal randomly weighted sums of widely orthant-dependent random variables with applications, Communications in Statistics - Theory and Methods, 50:4, 763-791, DOI: 10.1080/03610926.2019.1640879.

Baum, L. E., and M. Katz (1965),Convergence rates in the law of large numbers. Transactions of the American Mathematical Society 120 (1):10823. DOI:10.2307/1994170.

P. L. Hsu, H. Robbins (1947) Complete convergence and the law of large numbers, P. National Acad. Sci. USA, 33, 2531. DOI:10.1073/pnas.33.2.25.

Ding, Y., Y. Wu, S. Ma, X. Tao, and X. Wang (2017), Complete convergence and complete moment convergence for widely orthant-dependent random variables. Communication in Statistics-Theory and Methods 46 (16):827894. DOI:10.1080/03610926.2016.1177085.

Shen AT, Zhang Y, Volodin A (2015) Applications of the Rosenthal-type inequality for negatively superadditive dependent random variables. Metrika. 2015;78:295311.

Chao Lu, Xiaoqin Li, Rui Wang Xuejun Wang (2020), Complete and complete moment convergence for randomly weighted sums of ρ^* - mixing random variables and its applications, Statistics, 54:1, 205-237, DOI: 10.1080/02331888.2019.1707201.

Chen PY, Sung SH (2018), On complete convergence and complete moment convergence for weighted sums of ρ^*-mixing random variables. J Inequ Appl. 2018;2018:Article ID 121, 16 pages.