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P-Bifurcations of a Black-Scholes equation with Vasicek return rate model

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Abstract

This paper analyzes a Black-Scholes-Merton model with a stochastic return rate governed by the Vasicek model. Using a Taylor expansion approximation to solve the Fokker-Planck equation, we show that P-bifurcations occur in the solution whenever the return rate crosses its long-term mean. Numerical simulations support these findings.

Keywords: Black-Scholes-Merton model, Vasicek model, Fokker-Planck equation, P-bifurcation.
Mathematics Subject Classification (2020): 37H20, 35Q84, 62P05.

1 Introduction

A common critique of the Black-Scholes-Merton model is its assumption of constant coefficients, particularly the rate of return and volatility. While recent research has focused on extending the model to scenarios where volatility follows an Ito process, and the impact of random market movements on investment decisions Samimia and Mehrdoust (2024); Orlando et al. (2020), the effect of stochastic changes in expected returns has received less attention. This article explores a Black-Scholes-Merton-based model with the return rate of Vasicek form, and examines the implications of this randomness on the model's stability.

The p-bifurcation, in the context of stochastic differential equations (SDEs), describes a qualitative change in the stationary probability distribution of the system's response as a parameter crosses a critical value. More specifically, it signifies a change in the shape or support of the probability density function (PDF) or the invariant measure, potentially leading to the emergence of new stable states or the disappearance of existing ones. Unlike deterministic bifurcations that focus on the stability of fixed points, p-bifurcations are concerned with the changes in the statistical properties of the solution. This concept is crucial for understanding the long-term behavior of stochastic dynamical systems and their sensitivity to parameter variations (see for example Luo and Guo (2015); Tang et al. (2023); Failla and Pirrotta (2012)).

Here we consider the model:

$$\begin{aligned} dS_t &= \alpha\, R_t S_t\, dt + \sigma\, S_t dW_1(t) \\ dR_t &= (a - b\, R_t)dt + \epsilon\, dW_2(t). \end{aligned} \tag{1}$$

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The first equation is the Black-Scholes-Merton model for the underlying asset price, S_t; and the second equation is the Vasicek model for the return rate, R_t. The parameter $\alpha > 0$ quantifies the effect of R_t on S_t, $\sigma > 0$ is the volatility, and $a, b, \epsilon \geq 0$ are the Vasicek model parameters. The Vasicek model is typically used when the return rate of the underlying asset fluctuates over time and can be negative, which is relevant in markets where investors anticipate potential negative returns. Finally, $W_1(t)$ and $W_2(t)$ are independent standard Winner processes. The term $(a - b\,R_t)$ in the Vasicek model represents a mean-reverting force, pulling the return rate R_t towards its long-term average level a/b at a rate of b. The parameter ϵ controls stochastic fluctuations of the return rate around its expected value a/b. These models are also useful for pricing options or other derivatives where interest rate risk is significant, as they explicitly account for the stochastic nature of interest rates. The results of the paper is summarized as follows

Theorem 1.1. *As $t > 0$ increases, the dynamics of system (1) exhibit a P-bifurcation whenever $R(t)$ passes $R_0 = a/b$.*

We will first introduce the dimensionless model, its implementation, and the corresponding Fokker-Planck equation used to analyze the device. Then, we will present a method, based on the Taylor expansion of the Fokker-Planck equation's solutions with respect to the Black-Scholes-Merton model parameters, to investigate P-bifurcations and, consequently, the model's P-stability. Finally, we will present numerical simulations to validate our findings.

2 Fokker-Planck equation

First, with the change of variables and parameters defined by the rescaling

$$\bar{t} = bt, \quad \bar{S}_{\bar{t}} = S_t, \quad \bar{R}_{\bar{t}} = \frac{\alpha}{b} R_t, \quad \bar{\sigma} = \frac{\sigma}{\sqrt{b}}, \quad \bar{a} = \frac{\alpha a}{b^2},$$

$$\bar{\epsilon} = \frac{\epsilon\alpha}{b\sqrt{b}}, \quad \bar{W}_1(\bar{t}) = \frac{1}{\sqrt{b}} W_1(\bar{t}/b), \quad \bar{W}_2(\bar{t}) = \frac{1}{\sqrt{b}} W_2(\bar{t}/b)$$

equations (1), after dropping the bars and change the order of equations, becomes the dimensionless system:

$$\begin{aligned} dR_t &= (a - R_t)dt + \epsilon\, dW_2(t) \\ dS_t &= R_t S_t\, dt + \sigma S_t dW_1(t). \end{aligned} \tag{2}$$

The corresponding Fokker-Planck equation is then:

$$\frac{\partial p}{\partial t} = (\sigma^2 + 1 - r)p + s(2\sigma^2 - r)\frac{\partial p}{\partial s} + (a - r)\frac{\partial p}{\partial r} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 p}{\partial s^2} + \frac{\epsilon^2}{2}\frac{\partial^2 p}{\partial r^2}. \tag{3}$$

The equation lacks an explicit solution due to its coefficients, and also approximation algorithms, such as fast Fourier transform, are not suitable because of the polynomial coefficients; thus we must try other approximation methods. To over come this problem, we write the equation in the form, consider the right-hand side of the equation, which is the following elliptic second-order equation:

$$(1 - r)p - sr\frac{\partial p}{\partial s} + (a - r)\frac{\partial p}{\partial r} + \sigma^2\left(\frac{s^2}{2}\frac{\partial^2 p}{\partial s^2} + 2s\frac{\partial p}{\partial s} + p\right) + \frac{\epsilon^2}{2}\frac{\partial^2 p}{\partial r^2}.$$

Let $D \subseteq \mathbb{R}^{2,\geq 0}$ be a domain and $\mathcal{C}$ stands for smooth functions $p(t, s, r)$ defined on $\mathbb{R}^{\geq 0} \times D$, and define the linear operators $\mathcal{L} : \mathcal{C} \to \mathcal{C}$ and $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ as follows:

$$\mathcal{L} = (1 - r)I - sr\frac{\partial}{\partial s} + (a - r)\frac{\partial}{\partial r}, \qquad \mathcal{T} = \left(\frac{s^2}{2}\frac{\partial^2}{\partial s^2} + 2s\frac{\partial}{\partial s} + Id\right),$$

then for $p \in \mathcal{C}$ and $(s, r) \in D$, Eq. (3) is written by

$$\frac{\partial p}{\partial t} = \mathcal{L}p + \sigma^2\mathcal{T}p + \epsilon^2\frac{\partial^2}{2\partial r^2}p. \tag{4}$$

Considering (4) and expanding the solution $p(t,s,r)$ with respect to σ and ϵ, after replacing the expansion in (4), we obtain:

$$p(t,r,s) = p_0(t,r,s) + \sigma^2 p_1(t,r,s) + \epsilon^2 p_2(t,r,s) + O(4), \tag{5}$$

where p_0, p_1 and p_2 are the solutions of the following equations respectively:

$$\frac{\partial p_0}{\partial t} = \mathcal{L}p_0, \tag{6}$$

$$\frac{\partial p_1}{\partial t} = \mathcal{L}p_1 + \mathcal{T}p_0, \tag{7}$$

$$\frac{\partial p_2}{\partial t} = \mathcal{L}p_2 + \frac{\partial}{2\partial r^2}p_0. \tag{8}$$

Using the Lagrange method, the corresponding characteristic equation of (6) is as follows:

$$dt = \frac{ds}{rs} = \frac{dr}{a-r} = \frac{dp_0}{(1-r)p_0}$$

which yields

$$t(r) = c_1 - \ln|a-r|, \quad s(r) = \frac{c_2 e^{-r}}{|a-r|^a}, \quad p_0(r) = c_3|a-r|^{(a-1)}e^r.$$

Its easy to see that the above functions are linearly independent and therefore, the general solution of (6) is given, for any C^2 function $G_0(x,y)$, by

$$p_0(t,r,s) = |a-r|^{(a-1)}e^r G_0\Big(\ln|a-r| + t, |a-r|^a e^r s\Big).$$

Similarly, the corresponding characteristic equation of (7) is as follows:

$$dt = \frac{ds}{rs} = \frac{dr}{a-r} = \frac{dp_1}{(1-r)p_1 + \mathcal{T}p_0}$$

which yields

$$t(r) = c_1 - \ln|a-r|, \quad s(r) = \frac{c_2 e^{-r}}{|a-r|^a}, \quad p_1(r) = |a-r|^{(a-1)}e^r\Big(c_3 + \mathcal{U}(r)\Big),$$

where

$$\mathcal{U}(r) = \int |a-r|^{(1-a)}e^{-r}\frac{\mathcal{T}p_0\big(t(r),r,s(r)\big)}{a-r}\,dr.$$

Its easy to see that the above functions are linearly independent and therefore, the general solution of (6) is given, for any C^2 function $G_1(x,y)$, by

$$p_1(t,r,s) = |a-r|^{(a-1)}e^r G_1\Big(\ln|a-r| + t, |a-r|^a e^r s\Big) + |a-r|^{(a-1)}e^r\mathcal{U}(r).$$

The same argument shows that the general solution of (8) is given, for any C^2 function $G_2(x,y)$, by

$$p_2(t,r,s) = |a-r|^{(a-1)}e^r G_2\Big(\ln|a-r| + t, |a-r|^a e^r s\Big) + |a-r|^{(a-1)}e^r\mathcal{V}(r),$$

where

$$\mathcal{V}(r) = \int \left(\frac{|a-r|^{(1-a)}e^{-r}}{2(a-r)}\right)\frac{\partial^2 p_0}{\partial r^2}\big(t(r),r,s(r)\big)dr.$$

Replacing in (5), finally we find the solution of (4) as

$$p(t,r,s) = |a-r|^{(a-1)}e^r\left[\hat{G}_0(t,r,s) + \sigma^2\left(\hat{G}_1(t,r,s) + \mathcal{U}(r)\right) + \epsilon^2\left(\hat{G}_2(t,r,s) + \mathcal{V}(r)\right)\right] + O(4), \tag{9}$$

where $\hat{G}_i(t,r,s) = G_i\Big(\ln|a-r| + t, |a-r|^a e^r s\Big)$, $i = 0,1,2$. Let $\delta(r,s)$ be the Dirac delta function defined as

$$\delta(r,s) = \begin{cases} +\infty & s = r = 0 \\ 0 & \text{otherwise} \end{cases}, \text{ with } \int_{\mathbb{R}^2}\delta(r,s)dsdr = 1.$$

The solution to (4) with initial condition $p(0, R_0, S_0) = \delta(r - R_0, s - S_0)$ is the density function of the solution $(R(t), S(t))$ of (2) with initial condition $(R(0), S(0)) = (R_0, S_0)$. Consequently, changes in its behavior indicate a p-bifurcation of (2), and thus of (1) (see Risken (1996); Barkai (2001)). Due to computational difficulties associated with the Dirac delta function, the solution of (4) is often considered with the initial condition

$$p(0, r, s) = M(r, s) := \frac{m}{2\pi} \exp\left[-m\left(\frac{(s - S_0)^2 + (r - R_0)^2}{2}\right)\right],$$

where m is assumed big enough (indeed, we may assume $m \to +\infty$). Replacing the initial condition in (9), considering that the initial condition is σ, ϵ independent, we obtain

$$\hat{G}_0(0, r, s) = |a - r|^{(1-a)} e^{-r} M(r, s), \quad \hat{G}_1(0, r, s) = -\mathcal{U}(r), \quad \hat{G}_2(0, r, s) = -\mathcal{V}(r).$$

Considering $\hat{G}_0(0, r, s) = G_0(\ln|a - r|, |a - r|^a e^r s)$, the first above equality implies

$$G_0(r, s) = \begin{cases} \exp\left[r - a(1 + r) - e^r\right] M\left(a + e^r, s \exp[-a(1 + r) - e^r]\right) & a < r, \\ \exp\left[r - a(1 + r) + e^r\right] M\left(a - e^r, s \exp[-a(1 + r) + e^r]\right) & a > r. \end{cases}$$

and therefore,

$$\begin{aligned} \hat{G}_0(t, r, s) &= G_0(\ln|a - r| + t, |a - r|^a e^r s) \\ &= e^{e^t(a-r)-a} e^{(1-a)t} |a - r|^{1-a} M\left(a - e^t(a - r), s \exp\left[(e^t - 1)(a - r) - at\right]\right). \end{aligned}$$

Similarly, we can conclude

$$\begin{aligned} \hat{G}_1(t, r, s) &= e^{e^t(a-r)-a} e^{(1-a)t} |a - r|^{1-a} \mathcal{U}\left(a - e^t(a - r)\right), \\ \hat{G}_2(t, r, s) &= e^{e^t(a-r)-a} e^{(1-a)t} |a - r|^{1-a} \mathcal{V}\left(a - e^t(a - r)\right). \end{aligned}$$

Finally, replacing in (5) we obtain:

$$\begin{aligned} p(t, r, s) &= \underbrace{e^{(1-a)t} e^{(r-a)\left(1-e^t\right)} M\left(a - e^t(a - r), s \exp\left[(e^t - 1)(a - r) - at\right]\right)}_{=p_0(t,r,s)} \\ &+ \sigma^2 \underbrace{e^{(1-a)t} e^{(r-a)\left(1-e^t\right)} \left(\mathcal{U}(r) - \mathcal{U}(a - e^t(a - r))\right)}_{=p_1(t,r,s)} \\ &+ \epsilon^2 \underbrace{e^{(1-a)t} e^{(r-a)\left(1-e^t\right)} \left(\mathcal{V}(r) - \mathcal{V}(a - e^t(a - r))\right)}_{=p_2(t,r,s)} + O(4). \end{aligned} \tag{10}$$

Proof of theorem 1.1: It can be checked directly that, as $t \to +\infty$, we have $p_0(t, r, s)$ converged to zero. Given the integrability of $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{R}$, we have $\mathcal{U}(\pm\infty) = \mathcal{V}(\pm\infty) = 0$. Hence, $\mathcal{U}(a - e^t(a - r))$ and $\mathcal{V}(a - e^t(a - r))$ tend to zero as $t \to +\infty$. Consequently, $p_1(t, r, s)$ and $p_2(t, r, s)$ approach zero if $a < r$, and infinity if $a > r$, as $t \to +\infty$. Therefore, in the dimensional system (1), a P-bifurcation occurs in the solutions' dynamics when $t > 0$ increases and $R(t)$ crosses $R_0 = a/b$. □

3 Numerical simulation

Here we consider (2) with the parameters as Tabel 1. Fig. 1 shows three sample solutions of (2) for $0 \le t \le 20$, as we can see $R(t)$ intersects its eventual expected values $E\left(R(+\infty)\right) = 1$ several times.

Table 1: Parameter values in (2) and (10).

a	ϵ	σ	m	R_0	S_0
1	0.01	0.01	2100	0.8	1

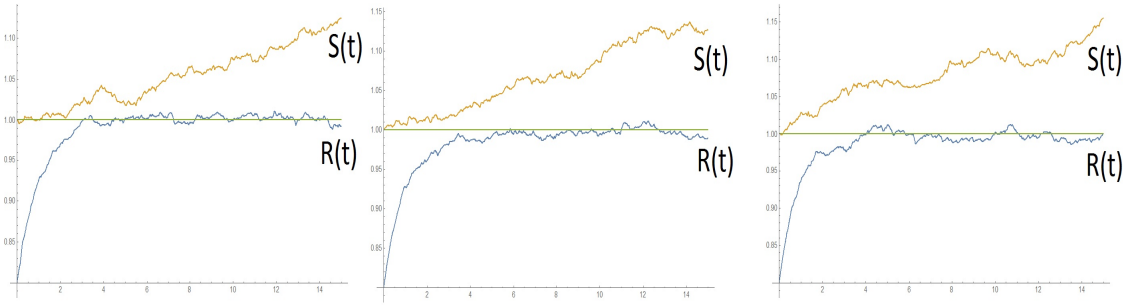


Figure 1: Sample solutions of (2)

Fig. 2 shows respectively from left to right $p(t, 0.999, 1)$ and $p(t, 1.1, 1)$ for $0 \leq t \leq 15$. We can see that for $r = 0.999 < 1 = a$, the density function p tends to infinity as t increases; and for $r = 1.1 > 1 = a$, the density function p tends to zero as t increases.

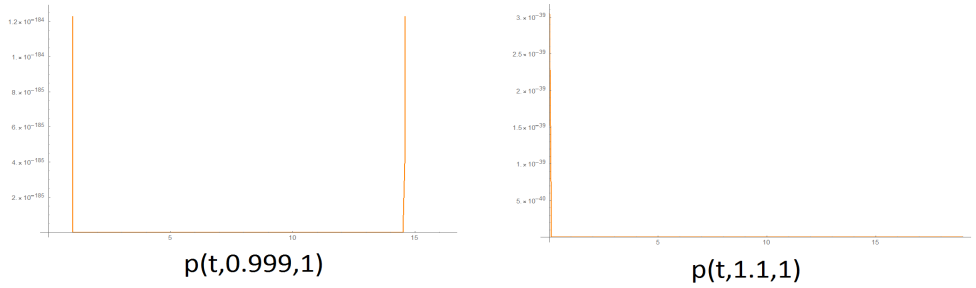


Figure 2: Plot of $p(t, 0.999, 1)$ and $p(t, 1.1, 1)$ for $0 \leq t \leq 15$

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