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The PRGM Prediction for Finite Population Parameters Using a Parametric Model Under SE and RN Loss Functions

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Abstract

Predicting finite population parameters is a fundamental issue in statistical sampling. This paper investigates the prediction of a general linear combination of finite population values under a normal superpopulation model using Bayesian and robust Bayesian approaches. The primary focus is on the Posterior Regret Gamma-Minimax (PRGM) predictor under two loss functions: Squared Error (SE) and Reflected Normal (RN). The differences between these loss functions and their impacts on prediction accuracy are examined. Using real data and simulations, the performance of PRGM, Bayesian, and frequentist predictors in predicting the finite population mean is compared. Results indicate that the PRGM predictor performs well under both loss functions, particularly when the prior distribution's variance is small.

Keywords: Bayesian approach, PRGM predictor, Finite population, SE loss function, RN loss function, Superpopulation model.

1 Introduction

In finite population sampling analysis, predicting parameters such as the population mean or total is of great importance. Classical methods have been developed based on unbiased estimators (e.g., Godambe (1955); Godambe and Joshi (1965)), but these approaches rely on strong assumptions regarding the data model and underlying parameters. In contrast, model-based Bayesian (superpopulation) approaches consider the finite population as a realization from a parametric model (e.g., the normal distribution) and provide predictions for population parameters based on that model. This framework allows greater flexibility in modeling uncertainty and incorporating prior knowledge.

However, classical Bayesian approaches require the specification of a particular prior distribution, which may lead to unstable results if there is uncertainty about the hyperparameters. To address this issue, robust Bayesian methods have been proposed. These methods define a

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class Γ of plausible prior distributions and aim to limit the impact of prior uncertainty on the resulting inferences. For instance, Berger (1984) emphasized the advantage of using a broad class of priors instead of a single prior distribution. In this framework, a set of Bayesian estimators is obtained corresponding to the members of Γ, and the goal is to select an estimator that yields optimal stability in prediction.

One prominent approach in this context is the Posterior Regret Gamma-minimax (PRGM) method, which optimizes predictions by minimizing the maximum "posterior regret." In other words, the PRGM predictor is the estimator that minimizes the worst-case difference between its posterior loss and that of the best possible predictor. The PRGM concept was first introduced by Rios Insua et al. (1995) and has since been applied to various Bayesian prediction problems. In the present study, the PRGM method is employed to predict parameters of a finite population.

The analysis assumes that the finite population arises from a superpopulation normal model. Using this model, PRGM predictors are derived for population parameters such as the mean or total. To evaluate prediction error, two common loss functions are considered: the Squared Error (SE) loss and the Reflected Normal (RN) loss. The SE loss is a standard criterion for assessing prediction error variance, while the RN loss is used to measure relative errors.

The main innovation of this study lies in the analytical derivation of PRGM predictors for finite population parameters under a heterogeneous normal superpopulation model and in the empirical comparison of this method with conventional Bayesian predictors. Through extensive real data and simulation studies, the performance of the PRGM predictor is compared with that of standard Bayesian predictors. The findings indicate that under prior uncertainty, the robust PRGM predictor generally yields lower prediction errors (under SE or RN loss) compared to traditional Bayesian methods.

In the related literature, several key studies have laid the theoretical foundation for this topic. For example, Godambe (1955) and Godambe and Joshi (1965) established the theoretical basis for Bayesian estimation in finite populations. Berger (1984) introduced the robust Bayesian framework and emphasized the importance of considering a class Γ of prior distributions. Bolfarine (1990) focused on linear Bayesian prediction in finite populations, and Ghosh and Kim (1993) proposed a robust predictor based on posterior regret considerations. Finally, Rios Insua et al. (1995) developed the PRGM method for Bayesian estimation. The following references represent the most important contributions to this area of study. Further developments by Jafaraghaie and Nematollahi (2016, 2018), Ghosh (2008), Si et al. (2015), and Kaminska and Porosinski (2009) have contributed to advancing both theoretical understanding and practical application of Bayesian prediction in finite population settings.

The paper is organized as follows: Section 2 defines the superpopulation model and the SE and RN loss functions. Section 3 presents the Bayesian and frequentist predictors, setting the stage for robust methods. Section 4 derives the PRGM predictor under SE and RN loss functions, emphasizing its robustness properties. Section 5 analyzes real data and simulation results to compare predictor performance, and Section 6 discusses the implications and future directions of this work.

2 Concepts and Definitions

2.1 Superpopulation Model

Consider a finite population $\mathcal{U} = \{1, 2, \ldots, N\}$ with unknown values represented by the vector $\mathbf{y} = (y_1, \ldots, y_N)^T$. A sample of size n, denoted $\mathbf{y}_s = \{y_i : i \in s\}$, is drawn from this population. The goal is to predict

$$\gamma(\mathbf{y}) = \sum_{i=1}^{N} p_i y_i,$$

where $p_i > 0$ are known weights. The superpopulation model is defined as:

$$y_i = \beta x_i + \varepsilon_i, \quad \varepsilon_i \sim N(0, \sigma^2 x_i), \quad i = 1, \ldots, N, \tag{1}$$

where x_i is an auxiliary variable, β is an unknown parameter, and σ^2 is a known variance. This model assumes that the target variable is, on average, proportional to the auxiliary variable.

2.2 Loss Functions

The loss function evaluates the accuracy of the predictor $\delta(\mathbf{y}_s)$ relative to $\gamma(\mathbf{y})$. This paper considers two loss functions:

1. **Squared Error (SE) Loss:** This is an unbounded symmetric loss function defined as:

   $$L(\Delta) = \Delta^2, \quad \Delta = \delta(\mathbf{y}_s) - \gamma(\mathbf{y}).$$

   Due to its simplicity and mathematical properties, SE is widely used but may be unsuitable in financial applications where large losses are undesirable.

2. **Reflected Normal (RN) Loss:** This is a bounded symmetric loss function defined as:

   $$L(\Delta) = 1 - e^{-\frac{\Delta^2}{2\gamma^2}},$$

   where γ is a shape parameter and $k = 1$ is the maximum loss. The bounded nature of RN makes it more suitable for financial and decision-making contexts where large losses are unacceptable.

Differences between SE and RN: The SE loss grows quadratically with error, heavily penalizing large errors, which is useful when both small and large errors are equally important. In contrast, the RN loss caps the penalty for large errors, making it suitable for scenarios where prediction stability is prioritized over absolute accuracy. However, RN may be less sensitive in cases where high precision is critical.

3 Bayesian and Frequentist Predictors

3.1 Frequentist Predictor

In the frequentist approach, the prediction of $\gamma(\mathbf{y})$ is based on the model (1) and the minimum variance unbiased estimator $\hat{\beta} = \frac{\bar{y}_s}{\bar{x}_s}$. The frequentist predictor is:

$$\hat{\gamma}(\mathbf{y}) = \sum_{i \in s} p_i y_i + \sum_{i \notin s} p_i x_i \hat{\beta}.$$

Due to its lack of prior information, this predictor may exhibit lower accuracy in small samples.

3.2 Bayesian Predictor

In the Bayesian approach, β is assumed to follow a normal prior distribution $N(\mu, \tau_0^2)$. The conditional posterior distribution of $\gamma(\mathbf{y})$ given the observed data $\mathbf{y}_s, \mathbf{x}_s$ is:

$$N\left(\sum_{i \in s} p_i y_i + \left(\frac{\bar{y}_s}{\bar{x}_s} + B_0\left(\mu - \frac{\bar{y}_s}{\bar{x}_s}\right)\right) h_s, \sigma^2\left(\sum_{i \notin s} p_i^2 x_i + (1 - B_0)\frac{h_s^2}{n\bar{x}_s}\right)\right),$$

where $B_0 = \frac{\sigma^2}{\sigma^2 + \tau_0^2 n \bar{x}_s}$ and $h_s = \sum_{i \notin s} p_i x_i$. The Bayesian predictor under both SE and RN loss functions is:

$$\delta^\pi = E(\gamma(\mathbf{y}) \mid \mathbf{y}_s, \mathbf{x}_s) = \sum_{i \in s} p_i y_i + \left(\frac{\bar{y}_s}{\bar{x}_s} + B_0\left(\mu - \frac{\bar{y}_s}{\bar{x}_s}\right)\right) h_s.$$

The posterior risk under SE and RN is computed as:

$$\rho(\pi, \delta^\pi) = \mathrm{Var}(\gamma(\mathbf{y}) \mid \mathbf{x}_s, \mathbf{y}_s), \quad \rho(\pi, \delta^\pi) = 1 - \frac{\gamma}{\sqrt{\gamma^2 + w}},$$

where $w = \mathrm{Var}(\gamma(\mathbf{y}) \mid \mathbf{x}_s, \mathbf{y}_s)$.

4 PRGM Predictor under SE and RN Loss Functions

In the robust Bayesian approach, the prior distribution (π) belongs to a class $\Gamma_\mu = \{\pi : \pi \sim N(\mu, \tau_0^2), \mu_1 \leq \mu \leq \mu_2\}$. The PRGM predictor is defined as:

$$\delta_\Gamma^{\text{PRGM}} = \frac{\delta^{\pi_{\mu_1}} + \delta^{\pi_{\mu_2}}}{2},$$

where $\delta^{\pi_{\mu_i}}$ is the Bayesian predictor under the prior $N(\mu_i, \tau_0^2)$. Under model (1) and the class Γ_μ, the PRGM predictor is:

$$\delta_\Gamma^{\text{PRGM}} = \sum_{i \in s} p_i y_i + \left(\frac{\bar{y}_s}{\bar{x}_s} + B_0 \left(\mu^* - \frac{\bar{y}_s}{\bar{x}_s} \right) \right) h_s, \quad \mu^* = \frac{\mu_1 + \mu_2}{2}.$$

Under both SE and RN loss functions, the PRGM predictor remains identical, as these loss functions depend on minimizing the conditional expected value.

5 Real Data and Simulation

5.1 Real Data

To evaluate predictor performance, real data from the Household Survey Statistics (HSS) were used, examining the relationship between income (auxiliary variable x_i) and net worth (target variable y_i). Analysis of variance indicated that income significantly affects net worth ($p-value < 0.05$). Model coefficients are presented in Table 1. Maximum likelihood estimates of the parameters are shown in Table 2.

Table 1: Test of Model Coefficients

	Estimate	Standard Error	T-Value	p-value
Intercept	-0.7726	0.1286	-6.007	0.250
X	4.5432	0.1996	22.765	0.000

Table 2: Maximum Likelihood Parameter Estimates

Parameters	Estimate	Standard Error
β	3.4363	0.066
σ^2	0.5320	0.0359

Table 3: The analysis of variance

Source of Variation	Sum of Squares	df	Mean Squares	F	p-value
Model	1437.2	1	1437.2	1797	0.000
Error	233.6	292	0.8	–	–
Total	1670.8	293	–	–	–

From Table 3, income is significant in the model and the coefficient of determination ($R^2 = 0.8602$) indicates that approximately 86% of the variation in net worth is explained by income.

5.2 Simulation

To compare the PRGM, Bayesian, and frequentist predictors, a simulation study was conducted using HSS data. Samples of sizes $n = 20, 30, 50$ were drawn from the superpopulation

model, with the process repeated 10,000 times. The Bayesian predictor used a prior $N(6, 0.01)$, the PRGM predictor used the class $\Gamma_\mu = \{N(\mu, \tau_0^2) : \mu \in [0, 6]\}$, and the frequentist predictor was computed.

Table 4: Predicted Finite Population Mean under SE and RN Loss Functions

Predictor	Predicted Value
δ^{π_0} (Bayes predictor)	1.66046
δ^{PRGM}	1.41057
$\delta^{\text{frequentist}}$	1.4518

Table 4 shows that the PRGM predictor is closest to the true mean (1.427406).

5.2.1 Performance Evaluation:

The Estimated Mean Squared Error (EMSE) and Estimated Absolute Bias (EAB) under SE, and the Estimated Risk Function (ERF) and Estimated Absolute Risk Bias (EARB) under RN, were computed. Results are presented in Tables 5 and 6.

Table 5: Simulated MSE and absolute bias of predictors for different τ_0^2, μ_0, and $\mu \in [1, 5]$ over Γ_μ (SE loss function)

	n	$\delta^\pi(\mu_0 = 1)$	$\delta^\pi(\mu_0 = 2)$	$\delta^\pi(\mu_0 = 4)$	$\delta^\pi(\mu_0 = 5)$	δ^{robust}	$\delta^{frequentist}$
				$\sigma_0^2 = 0.001$			
EMSE	20	0.01008	0.01003	0.01004	0.01009	0.01002	0.01020
	30	0.00650	0.00640	0.00640	0.00650	0.00637	0.00654
	50	0.00380	0.00352	0.00341	0.00356	0.00339	0.00352
EAB	20	0.07890	0.07880	0.07902	0.07935	0.07884	0.07961
	30	0.06363	0.06321	0.06336	0.06396	0.06311	0.06401
	50	0.04889	0.04707	0.04643	0.04765	0.04624	0.04714
				$\sigma_0^2 = 0.01$			
EMSE	20	0.03076	0.01232	0.00912	0.01406	0.00888	0.01020
	30	0.01915	0.01126	0.00602	0.01199	0.00556	0.00654
	50	0.01627	0.00994	0.00397	0.01131	0.00339	0.00352
EAB	20	0.16806	0.10044	0.07584	0.10719	0.07367	0.07961
	30	0.12174	0.08379	0.06217	0.08922	0.05867	0.06401
	50	0.10423	0.08186	0.05147	0.08916	0.04653	0.04714
				$\sigma_0^2 = 0.1$			
EMSE	20	0.35610	0.12434	0.01943	0.14629	0.01211	0.01020
	30	0.29285	0.10291	0.01695	0.12092	0.01095	0.00654
	50	0.22269	0.07993	0.01455	0.09193	0.01055	0.00352
EAB	20	0.59601	0.35168	0.13697	0.38129	0.10736	0.07961
	30	0.53853	0.31803	0.12345	0.34349	0.09765	0.06401
	50	0.46581	0.27622	0.10845	0.29300	0.08918	0.04714

5.3 Analysis of Results

The PRGM predictor outperforms Bayesian and frequentist predictors under both SE and RN loss functions, particularly when τ_0^2 is small. The RN loss, due to its bounded nature, performs better in scenarios where prediction stability is critical, but it may be slightly less accurate in terms of absolute precision. As sample size increases, EMSE, EAB, ERF, and EARB decrease.

Table 6: Simulated risk function (EMSE) and absolute risk bias (EAB) for different λ_0 and σ_0^2 over Γ_μ (RN loss function)

	n	$\delta^\pi(\mu_0=1)$	$\delta^\pi(\mu_0=2)$	$\delta^\pi(\mu_0=4)$	$\delta^\pi(\mu_0=5)$	δ^{robust}	$\delta^{frequentist}$
$\sigma_0^2=0.001$							
EMSE	20	0.01948	0.01940	0.01942	0.01952	0.01938	0.01972
	30	0.01273	0.01255	0.01255	0.01274	0.01249	0.01282
	50	0.00750	0.00697	0.00674	0.00705	0.00671	0.00696
EAB	20	0.00532	0.00327	0.00083	0.00289	0.00128	0.00088
	30	0.00347	0.00219	0.00051	0.00183	0.00122	0.00032
	50	0.00310	0.00214	0.00037	0.00054	0.00082	0.00023
$\sigma_0^2=0.01$							
EMSE	20	0.05912	0.02414	0.01772	0.02746	0.01721	0.01972
	30	0.03694	0.02172	0.01183	0.02322	0.01092	0.01282
	50	0.03117	0.01937	0.00786	0.02205	0.00671	0.00696
EAB	20	0.03914	0.02407	0.00915	0.02541	0.00767	0.00088
	30	0.02808	0.01698	0.00630	0.01777	0.00541	0.00032
	50	0.02102	0.01289	0.00378	0.01203	0.00458	0.00023
$\sigma_0^2=0.1$							
EMSE	20	0.50823	0.21966	0.03803	0.25291	0.02388	0.01972
	30	0.43973	0.18489	0.03314	0.21285	0.02153	0.01282
	50	0.35339	0.14593	0.02840	0.16496	0.02063	0.00696
EAB	20	0.07459	0.06843	0.03285	0.07096	0.02614	0.00088
	30	0.07363	0.06436	0.02953	0.06684	0.02372	0.00032
	50	0.07302	0.05815	0.02473	0.05973	0.02093	0.00023

6 Discussion and Conclusion

This paper investigated the prediction of finite population parameters using the PRGM predictor under SE and RN loss functions. Simulation results demonstrated that the PRGM predictor outperforms Bayesian and frequentist predictors, especially with small prior variances. The RN loss function, due to its bounded nature, is more suitable for applications where large losses are unacceptable, while the SE loss excels in scenarios prioritizing high accuracy. Future research should explore these predictors under more complex models or non-normal data.

7 References

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