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Parameters Estimation for Two Lomax Exponential Distributions Under Joint Type-II Censored Data

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Abstract

In this article, we investigate parameter estimation for two Lomax exponential distributions under a joint Type-II censoring scheme. This distribution is a flexible model capable of representing lifetime data with both increasing and decreasing trends. Using the Expectation-Maximization algorithm, we derive the maximum likelihood estimators for the parameters. We also compute approximate confidence intervals based on the missing information matrix. To evaluate the performance of the proposed method, we conduct a Monte Carlo simulation study across various sample sizes and demonstrate its practical utility through the analysis of a real dataset.

Keywords: Expectation-Maximization algorithm, Lomax exponential distribution, Joint type-II censoring scheme.
Mathematics Subject Classification (2020): 62F10, 62F25, 62N02.

1 Introduction

In survival analysis, censoring occurs when only partial information about a units failure time is available due to study design or data limitations. The most important type of censored data is the Type-II censoring scheme, where events are monitored until a predetermined number occur, after which remaining subjects are censored (see Lawless (2003)). When comparing two populations from separate sources under identical conditions, two independent samples are taken, with sizes m and n, and subjected to a life-testing experiment. The experiment concludes after a specified number of failures, r, where $1 \leq r \leq N$ and $N = m + n$. This method allows for population comparison based on observed failures and is known as a joint Type-II censoring scheme.

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Previous research has extensively investigated joint Type-II censoring schemes and their applications. Notably, Balakrishnan and Rasouli (2008) developed exact inference methods for two exponential distributions under a joint Type-II censoring scheme. Further advancing this field, Akbari and *et al.* (2023) focused on statistical inference for two Burr-XII distribution using the same censoring framework. More recently, Qiu and Gui (2023) extended these methodologies by examining inference techniques for two Gumbel distributions under joint Type-II censoring scheme.

The Lomax Exponential (LE) distribution, introduced by Ijaz and *et al.* (2019), is a two-parameter modification of the classical Lomax distribution. This flexible model is particularly useful for analyzing lifetime data with both increasing and decreasing hazard rate patterns. Ijaz and *et al.* (2019) conducted a comprehensive study of the LE distribution statistical properties, examining its moments, entropy measures, hazard and survival functions, as well as its median, mode, and order statistics.

Despite these thorough investigations, research on the LE distribution under censoring schemes remains limited. Addressing this gap, Akbari and *et al.* (2024) recently developed maximum likelihood estimators (MLEs) and approximate confidence intervals (ACIs) for the LE distribution parameters under Type-II censoring scheme. Building on this work, we continue their research by examining joint type-II censored data from two LE populations. Section 2 presents the joint Type-II censoring scheme model for two LE populations. MLEs of the unknown parameters are derived using the Expectation-Maximization (EM) algorithm, and ACIs are computed based on the fisher information matrix. Additionally, a simulation study and a real data application are provided in Sections 3 and 4, respectively, to evaluate the performance of the estimation method.

2 Model specification and likelihood inferences

Let $\mathbf{X} = (X_1, \ldots, X_m)'$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)'$ denote the lifetime vectors of random samples from the first and second populations, respectively, where both populations follow a LE distribution. The probability density function f_i and survival function $\overline{F}_i$ $(i = 1, 2)$ for the $i-th$ population $(i = 1, 2)$ are given by

$$f_1(x; \theta_1, \theta_2) = \frac{\theta_1}{\theta_2}(1+x)e^x[1 + \frac{xe^x}{\theta_2}]^{-(\theta_1+1)}, x > 0, \quad \theta_1 > 0, \quad \theta_2 > 0, \tag{1}$$

$$\overline{F}_1(x; \theta_1, \theta_2) = [1 + \frac{xe^x}{\theta_2}]^{-\theta_1}, x > 0, \quad \theta_1 > 0, \quad \theta_2 > 0, \tag{2}$$

$$f_2(y; \theta_3, \theta_4) = \frac{\theta_3}{\theta_4}(1+y)e^y[1 + \frac{ye^y}{\theta_4}]^{-(\theta_3+1)}, y > 0, \quad \theta_3 > 0, \quad \theta_4 > 0 \tag{3}$$

$$\overline{F}_2(y; \theta_3, \theta_4) = [1 + \frac{ye^y}{\theta_4}]^{-\theta_3}, y > 0, \quad \theta_3 > 0, \quad \theta_4 > 0, \tag{4}$$

where θ_i $(i = 1, 3)$ and θ_j $(j = 2, 4)$ control the shape and scale parameters of the distribution, respectively. Consider $N = m + n$ as the total number of selected samples from the two populations, where r denotes the total number of failure or observed units. Let $W_1 < W_2 < \ldots < W_N$ represent the ordered statistics of the combined N units. Under the joint Type-II censoring scheme, the observed data is represented by the pair $(\mathbf{W}, \mathbf{S})$, where $\mathbf{W} = (W_1, W_2, \ldots, W_r)$ are the first r ordered failure times and $\mathbf{S} = (S_1, S_2, \ldots, S_r)$ is the indicator vector such that $S_i = 1$ if W_i belongs to X -failures and $S_i = 0$ if W_i belongs to Y -failures for $i = 1, 2, \ldots, r$. Let $M_r = \sum_{i=1}^{r} S_i$ and $N_r = \sum_{i=1}^{r} (1 - S_i) = r - M_r$ denote the number of failures from the X and Y samples, respectively. The likelihood function for the observed data based on the joint

density function of $(\mathbf{W}, \mathbf{S})$ is

$$L(\boldsymbol{\theta}; \mathbf{w}, \mathbf{s}) = \frac{m!n!}{(m-m_r)!(n-n_r)!} \prod_{i=1}^{r} [f_1(w_i; \theta_1, \theta_2)]^{s_i} [f_2(w_i; \theta_3, \theta_4)]^{1-s_i} \times [\overline{F}_1(w_r; \theta_1, \theta_2)]^{m-m_r} [\overline{F}_2(w_r; \theta_3, \theta_4)]^{n-n_r},$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3, \theta_4)'$ and f_i and $\overline{F}_i$ $(i = 1, 2)$ are defined in equations (1)-(4). Then logarithm of the likelihood function without the constant term is given by

$$\begin{aligned} l_1 = \ln L(\boldsymbol{\theta}; \mathbf{w}, \mathbf{s}) \propto & m_r \ln(\theta_1) + \theta_1 m \ln(\theta_2) + \sum_{i=1}^{r} \ln(w_i + 1) + \sum_{i=1}^{r} w_i + n_r \ln(\theta_3) + \theta_3 n \ln(\theta_4) \\ & - (\theta_1 + 1) \sum_{i=1}^{r} s_i \ln(\theta_2 + w_i e^{w_i}) - (\theta_3 + 1) \sum_{i=1}^{r} (1 - s_i) \ln(\theta_4 + w_i e^{w_i}) \\ & - \theta_1 (m - m_r) \ln(\theta_2 + w_r e^{w_r}) - \theta_3 (n - n_r) \ln(\theta_4 + w_r e^{w_r}). \end{aligned} \tag{5}$$

To obtain the MLEs of the parameters θ_i $(i = 1, 2, 3, 4)$, differentiation of equation (5) with respect to the parameters is performed. This leads to obtaining the likelihood equations in the following form

$$\frac{\partial l_1}{\partial \theta_1} = \frac{m_r}{\theta_1} + m \ln(\theta_2) - (m - m_r) \ln(\theta_2 + w_r e^{w_r}) - \sum_{i=1}^{r} s_i \ln(\theta_2 + w_i e^{w_i}) = 0, \tag{6}$$

$$\frac{\partial l_1}{\partial \theta_2} = \frac{m\theta_1}{\theta_2} - \frac{\theta_1 (m - m_r)}{\theta_2 + w_r e^{w_r}} - (\theta_1 + 1) \sum_{i=1}^{r} s_i (\theta_2 + w_i e^{w_i})^{-1} = 0, \tag{7}$$

$$\frac{\partial l_1}{\partial \theta_3} = \frac{n_r}{\theta_3} + n \ln(\theta_4) - (n - n_r) \ln(\theta_4 + w_r e^{w_r}) - \sum_{i=1}^{r} (1 - s_i) \ln(\theta_4 + w_i e^{w_i}) = 0, \tag{8}$$

$$\frac{\partial l_1}{\partial \theta_4} = \frac{n\theta_3}{\theta_4} - \frac{\theta_3 (n - n_r)}{\theta_4 + w_r e^{w_r}} - (\theta_3 + 1) \sum_{i=1}^{r} (1 - s_i) (\theta_4 + w_i e^{w_i})^{-1} = 0. \tag{9}$$

As closed-form solutions are unavailable to solve the above equations, numerical methods are required to compute the MLEs. The EM algorithm is used as an iterative estimation approach for censored data, with implementation details provided in Subsection 2.1.

2.1 The EM algorithm

The EM algorithm is particularly useful for parameter estimation in models with incomplete data, such as missing or censored observations, which commonly occur in real-world applications. The method was initially presented by Dempster and *et al.* (1977). This iterative method consists of two key steps: the expectation step (E-step) and the maximization step (M-step). During the E-step, the algorithm calculates the expected value of the censored variables based on the observed data, while in the M-step, it maximizes this expected value to determine the parameters. Censored data is displayed by $(\mathbf{Z}, \boldsymbol{\delta})$ where $\mathbf{Z} = (Z_1, \ldots, Z_{N-r})$ and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_{N-r})$ such that $\delta_j = 1$ and $\delta_j = 0$, if Z_j belongs to X and Y, respectively, where $j = 1, 2, \ldots, N - r$. It is clear that $\sum_{j=1}^{N-r} \delta_j = m - M_r$ and $\sum_{j=1}^{N-r} (1 - \delta_j) = n - N_r$. The combined observed and censored data are used to represent the completed dataset as $\boldsymbol{C} = (\mathbf{W}, \mathbf{S}, \mathbf{Z}, \boldsymbol{\delta})$. According to complete data, the maximum-likelihood function is given by

$$L_C(\boldsymbol{\theta}; \mathbf{w}, \mathbf{s}, \boldsymbol{\delta}, \mathbf{z}) = \prod_{i=1}^{r} [f_1(w_i; \theta_1, \theta_2)]^{s_i} [f_2(w_i; \theta_3, \theta_4)]^{1-s_i} \prod_{j=1}^{N-r} [f_1(z_j; \theta_1, \theta_2)]^{\delta_i} [f_2(z_j; \theta_3, \theta_4)]^{1-\delta_i}, \tag{10}$$

then, logarithm of the likelihood function based on equation (10) is as follows:

$$l_2 = \ln L_C(\boldsymbol{\theta}; \mathbf{w}, \mathbf{s}, \boldsymbol{\delta}, \mathbf{z}) = m\ln(\theta_1) + m\theta_1 \ln(\theta_2) + n\ln(\theta_3) + n\theta_3 \ln(\theta_4) + \sum_{i=1}^{r} \ln(w_i + 1)$$
$$+ \sum_{i=1}^{r} w_i + \sum_{j=1}^{N-r} \ln(z_j + 1) + \sum_{j=1}^{N-r} z_j - (\theta_1 + 1)\Big[\sum_{i=1}^{r} s_i \ln(\theta_2 + w_i e^{w_i}) + \sum_{j=1}^{N-r} \delta_j \ln(\theta_2 + z_j e^{z_j})\Big]$$
$$- (\theta_3 + 1)\Big[\sum_{i=1}^{r} (1 - s_i)\ln(\theta_4 + w_i e^{w_i}) + \sum_{j=1}^{N-r} (1 - \delta_j)\ln(\theta_4 + z_j e^{z_j})\Big]. \quad (11)$$

Following the E-step, the conditional expectations from derivatives of equation (11) with respect to the unknown parameters given the observed data are calculated as follows:

$$E\Big(\frac{\partial l_2}{\partial \theta_1}|z_j > w_r\Big) = \frac{m}{\theta_1} + m\ln(\theta_2) - \sum_{i=1}^{r} s_i \ln(\theta_2 + w_i e^{w_i})$$
$$- \sum_{j=1}^{N-r} \delta_j E\big[\ln(\theta_2 + z_j e^{z_j})|z_j > w_r\big] = 0, \quad (12)$$

$$E\Big(\frac{\partial l_2}{\partial \theta_2}|z_j > w_r\Big) = \frac{m\theta_1}{\theta_2} - (\theta_1 + 1)\Big[\sum_{i=1}^{r} \frac{s_i}{\theta_2 + w_i e^{w_i}} + \sum_{j=1}^{N-r} \delta_j E\Big(\frac{1}{\theta_2 + z_j e^{z_j}}|z_j > w_r\Big)\Big] = 0, \quad (13)$$

$$E\Big(\frac{\partial l_2}{\partial \theta_3}|z_j > w_r\Big) = \frac{n}{\theta_3} + n\ln(\theta_4) - \sum_{i=1}^{r} (1 - s_i)\ln(\theta_4 + w_i e^{w_i})$$
$$- \sum_{j=1}^{N-r} (1 - \delta_j) E\big[\ln(\theta_4 + z_j e^{z_j})|z_j > w_r\big] = 0, \quad (14)$$

$$E\Big(\frac{\partial l_2}{\partial \theta_4}|z_j > w_r\Big) = \frac{n\theta_3}{\theta_4} - (\theta_3 + 1)\Big[\sum_{i=1}^{r} \frac{(1 - s_i)}{\theta_4 + w_i e^{w_i}}$$
$$+ \sum_{j=1}^{N-r} (1 - \delta_j) E\Big(\frac{1}{\theta_4 + z_j e^{z_j}}|z_j > w_r\Big)\Big] = 0. \quad (15)$$

To compute the conditional expectations in equations (12)-(15), we require the conditional probability density function of $Z_j = z_j$ given $z_j > w_r$. This follows a left-truncated LE distribution (Ng and *et al.* (2002)), given by:

$$f_{Z_j|W_r}(z_j|z_j > w_r) = \left\{\frac{f_1(z_j; \theta_1, \theta_2)}{\overline{F}_1(w_r; \theta_1, \theta_2)}\right\}^{\delta_j} \left\{\frac{f_2(z_j; \theta_3, \theta_4)}{\overline{F}_2(w_r; \theta_3, \theta_4)}\right\}^{1-\delta_j}. \quad (16)$$

According to equation (16), conditional probability density function reduces to the first component if $\delta_j = 1$ and to the second component if $\delta_j = 0$. The following expectations can be calculated using equation (16) as follows:

$$E\big[\ln(\theta_2 + ze^{z})\big] = \ln(\theta_2 + w_r e^{w_r}) + \frac{1}{\theta_1}, \quad E\big[(\theta_2 + ze^{z})^{-1}\big] = \frac{\theta_1}{(\theta_1 + 1)(\theta_2 + w_r e^{w_r})},$$
$$E\big[\ln(\theta_4 + ze^{z})\big] = \ln(\theta_4 + w_r e^{w_r}) + \frac{1}{\theta_3}, \quad E\big[(\theta_4 + ze^{z})^{-1}\big] = \frac{\theta_3}{(\theta_3 + 1)(\theta_4 + w_r e^{w_r})}.$$

Based on the M- step, let $\hat{\theta}_i^h$ denote the MLEs of parameter θ_i $(i = 1, 2, 3, 4)$ at the h^{th} iteration. Then, the MLEs for the θ_i at the $(h+1)^{th}$ iteration are given by:

$$\widehat{\theta_1}^{(h+1)} = \frac{m_r}{-m\ln(\widehat{\theta_2}^{h}) + \sum_{i=1}^{r} s_i \ln(\widehat{\theta_2}^{h} + w_i e^{w_i}) + (m - m_r)\ln(\widehat{\theta_2}^{h} + w_r e^{w_r})}, \quad (17)$$

$$\widehat{\theta_2}^{(h+1)} = \frac{m\widehat{\theta_1}^{(h+1)}}{(\widehat{\theta_1}^{(h+1)} + 1)\sum_{i=1}^{r} s_i(\widehat{\theta_2}^{h} + w_i e^{w_i})^{-1} + (m - m_r)\widehat{\theta_1}^{(h+1)}\big[\widehat{\theta_2}^{h} + w_r e^{w_r}\big]^{-1}}, \quad (18)$$

$$\widehat{\theta}_3^{(h+1)} = \frac{n_r}{-n\ln(\widehat{\theta}_4^{\,h}) + \sum_{i=1}^{r}(1-s_i)\ln(\widehat{\theta}_4^{\,h} + w_i e^{w_i}) + (n-n_r)\ln(\widehat{\theta}_4^{\,h} + w_r e^{w_r})}, \tag{19}$$

$$\widehat{\theta}_4^{(h+1)} = \frac{n\widehat{\theta}_3^{(h+1)}}{(\widehat{\theta}_3^{(h+1)}+1)\sum_{i=1}^{r}(1-s_i)(\widehat{\theta}_4^{\,h} + w_i e^{w_i})^{-1} + (n-n_r)\widehat{\theta}_3^{(h+1)}\left[\widehat{\theta}_4^{\,h} + w_r e^{w_r}\right]^{-1}}. \tag{20}$$

The iteration process in equations (17)-(20) continues until the desired convergence is reached when $|\widehat{\theta}_1^{(h+1)} - \widehat{\theta}_1^{h}| + |\widehat{\theta}_2^{(h+1)} - \widehat{\theta}_2^{h}| + |\widehat{\theta}_3^{(h+1)} - \widehat{\theta}_3^{h}| + |\widehat{\theta}_4^{(h+1)} - \widehat{\theta}_4^{h}| < \varepsilon$, for a small value of $\varepsilon > 0$.

2.2 Approximation confidence interval estimation

This subsection presents the computation of the fisher information matrix, a crucial component for constructing the variance-covariance matrix needed to derive confidence intervals for the unknown parameters. Following the missing information principle introduced by Louis (1982), we formulate the observed information matrix as follows:

$$\mathbf{I}_{\mathbf{W},\mathbf{S}}(\boldsymbol{\theta}) = \mathbf{I}_C(\boldsymbol{\theta}) - \mathbf{I}_{\mathbf{Z},\boldsymbol{\delta}|\mathbf{W},\mathbf{S}}(\boldsymbol{\theta}), \tag{21}$$

where $\mathbf{I}_{\mathbf{W},\mathbf{S}}(\boldsymbol{\theta})$, $\mathbf{I}_C(\boldsymbol{\theta})$ and $\mathbf{I}_{\mathbf{Z},\boldsymbol{\delta}|\mathbf{W},\mathbf{S}}(\boldsymbol{\theta})$ are the observed, complete, and missing information matrices, respectively, and $\mathbf{C} = (X_1, ..., X_m, Y_1, ..., Y_n)$. Based on the LE distribution, the complete information matrix is presented in the following form

$$\mathbf{I}_C(\boldsymbol{\theta}) = \begin{bmatrix} \frac{m}{\theta_1^2} & -\frac{m}{\theta_2(\theta_1+1)} & 0 & 0 \\ -\frac{m}{\theta_2(\theta_1+1)} & \frac{m\theta_1}{\theta_2^2(\theta_1+2)} & 0 & 0 \\ 0 & 0 & \frac{n}{\theta_3} & -\frac{n}{\theta_4(\theta_3+1)} \\ 0 & 0 & -\frac{n}{\theta_4(\theta_3+1)} & \frac{n\theta_3}{\theta_4^2(\theta_3+2)} \end{bmatrix}, \tag{22}$$

and the missing information matrix can be computed as follows:

$$\mathbf{I}_{\mathbf{Z},\boldsymbol{\delta}|\mathbf{W},\mathbf{S}}(\boldsymbol{\theta}) = \begin{bmatrix} \frac{m-m_r}{\theta_1^2} & \frac{m_r-m}{(\theta_1+1)(\theta_2+w_r e^{w_r})} & 0 & 0 \\ \frac{m_r-m}{(\theta_1+1)(\theta_2+w_r e^{w_r})} & \frac{\theta_1(m-m_r)}{(\theta_1+2)(\theta_2+w_r e^{w_r})^2} & 0 & 0 \\ 0 & 0 & \frac{n-n_r}{\theta_3^2} & \frac{n_r-n}{(\theta_3+1)(\theta_4+w_r e^{w_r})} \\ 0 & 0 & \frac{n_r-n}{(\theta_3+1)(\theta_4+w_r e^{w_r})} & \frac{\theta_3(n-n_r)}{(\theta_3+2)(\theta_4+w_r e^{w_r})^2} \end{bmatrix}. \tag{23}$$

Using Relation 21 along with equations (22) and (23), we obtain the observed information matrix. A $(1-\alpha)100\%$ ACIs for θ_i is then constructed by inverting the observed information matrix $\mathbf{I}_{\mathbf{W},\mathbf{S}}(\widehat{\boldsymbol{\theta}})$ as $\widehat{\theta}_i \pm z_{\alpha/2}\sqrt{var(\widehat{\theta}_i)}$ $(i = 1, 2, 3, 4)$. $z_{\alpha/2}$ denotes the upper $\alpha/2$ quantile of the standard normal distribution, $\text{var}(\widehat{\theta}_i)$ is the i^{th} diagonal element of $\left(\mathbf{I}_{\mathbf{W},\mathbf{S}}(\widehat{\boldsymbol{\theta}})\right)^{-1}$ for $i = 1, 2, 3, 4$ and $\widehat{\boldsymbol{\theta}} = (\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3, \widehat{\theta}_4)'$ is the vector of MLEs of the vector $\boldsymbol{\theta}$.

3 Simulation Study

A Monte Carlo simulation study was conducted using the R program to assess the estimation technique (R Core Team (2020)). The simulation was repeated 1000 times (n_s =

10^3), and the data was generated from the LE$(x, 0.5, 1)$ and LE$(y, 0.8, 1.5)$ distributions, under various joint type II censoring schemes and different sample sizes, specified as (m, n, r) $=\{(25, 20, 25), (25, 20, 30), (40, 40, 25), (40, 40, 30)\}$. Here, m and n denote the sizes of the first and second samples respectively, and r is the censoring point.The MLEs of the parameters were calculated using the EM method. Additionally, the ACIs were calculated for the unknown parameters using the concept of the missing information matrix. Table 1 shows the bias and mean squared error (MSE) and Table 2 shows the upper and lower limits (L and U, respectively), and interval length (IL) for the parameters. The bias of the estimated parameters is determined as bias$(\widehat{\theta}_i) = \widehat{\theta}_i - \theta_i$, $(i = 1, 2, 3, 4)$ and the MSE is defined as

$$\text{MSE}(\widehat{\theta}_i) = \frac{1}{n_s}\sum_{j=1}^{n_s}(\widehat{\theta}_{i.j} - \theta_i)^2, \quad i = 1, 2, 3, 4.$$

According to Table 1, it is evident that EM estimations have demonstrated favorable performance in terms of bias and MSE. Moreover,based on the mentioned Tables, as the values of m, n, and r increase, there is a general reduction in bias, MSE and IL.

Table 1: Bias and MSE of EM estimates

		$\widehat{\theta}_1$		$\widehat{\theta}_2$		$\widehat{\theta}_3$		$\widehat{\theta}_4$	
(m, n)	r	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
$(25, 20)$	25	0.1415	0.0809	0.3675	1.0545	0.1272	0.1333	0.2650	1.4726
	30	0.1129	0.0616	0.3856	0.9116	0.0749	0.1051	0.1643	0.4734
$(40, 40)$	25	0.1057	0.0419	0.2510	0.1187	0.1125	0.0823	0.1104	0.0864
	30	0.1011	0.0352	0.1775	0.0670	0.0700	0.0436	0.1166	0.0479

Table 2: 95% L and U bound estimates and IL for parameters based on the EM method

		θ_1			θ_2			θ_3			θ_4		
(m, n)	r	L	U	IL	L	U	IL	L	U	IL	L	U	IL
$(25, 20)$	25	0.1193	1.1065	0.9872	0.3699	2.4013	2.0314	0.2379	1.5119	1.2740	0.6334	2.6952	2.0618
	30	0.1766	1.1065	0.9299	0.0000	2.8125	2.8125	0.4966	1.3577	0.8611	1.2736	2.2565	0.9829
$(40, 40)$	25	0.2798	0.9224	0.6426	0.4932	1.8617	1.3684	0.4768	1.2633	0.7866	1.1212	2.1121	0.9910
	30	0.3264	0.8849	0.5585	0.6998	1.8022	1.1024	0.5355	1.2896	0.7541	1.1983	2.0225	0.8242

4 Application to Real Data

In this section, in order to evaluate the presented estimation method, we use the real dataset presented by Ijaz and *et al.* (2019). The dataset displays the losses caused by wind catastrophes recorded in 1977, that we randomly divided this data into two groups. These data are presented in Table 3. The Kolmogorov-Smirnov (K-S) test was used to check whether two datasets follow the LE distribution using the "fitdistrplus" package in R-statistical language and the results are shown in Table 4. According to the p-values, the LE distribution fits well for both datasets. According to Table 5, it can be seen that by increasing the value of r, the bias and MSE decrease. Also, according to Table 6, by increasing the value of r, in generally, IL decreases and the obtained intervals include the real dataset.

Table 3: Losses due to wind catastrophes

Group	Data																		
X	27	2	17	5	2	32	43	3	4	2	2	6	6	4	3	2	2	2	2	24
Y	2	2	2	2	3	3	4	5	5	5	6	6	8	8	9	15	22	23	25	

Table 4: MLEs and K-S test results for real dataset

Data			K-S distance	p-value
X	$\widehat{\theta}_1$=0.1014	$\widehat{\theta}_2$=3.9103	0.27086	0.1063
Y	$\widehat{\theta}_3$=0.1318	$\widehat{\theta}_4$=11.7951	0.12695	0.9194

Table 5: Bias and MSE of estimations based on the EM method for real dataset

		$\widehat{\theta}_1$		$\widehat{\theta}_2$		$\widehat{\theta}_3$		$\widehat{\theta}_4$	
(m,n)	r	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
(20, 19)	31	-0.0014	0.0002	-0.0103	0.0003	-0.0318	0.0010	-0.0951	0.0090
	35	0.0006	0.0001	-0.0003	0.0002	-0.0018	0.0001	-0.0151	0.0002

Table 6: 95% L and U bound estimates and its IL based on the EM method for real dataset

		θ_1			θ_2			θ_3			θ_4		
(m,n)	r	L	U	IL	L	U	IL	L	U	IL	L	U	IL
(20, 19)	31	0.0607	0.1393	0.0786	0.0000	10.6097	10.6097	0.0000	0.2353	0.2353	0.0000	34.6607	34.6607
	35	0.0634	0.1406	0.0771	0.0000	10.5356	10.5356	0.0000	0.2656	0.2656	0.0000	30.9332	30.9332

Discussion and Results

This paper has investigated the estimation problem of the unknown parameters for two LE distributions under a joint type-II censoring scheme. Since there was no closed-form solution for the MLEs, the EM method was employed. ACIs were then calculated using the concept of the missing information matrix. The simulation results showed that MSE and IL, in generally, decrease with increasing m, n and r values. In addition, the presented method, was applied to the real dataset to verify the accuracy of the simulation results.

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