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From probability uncertainty to nonlinear PDEs

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Abstract

The problem of pricing a contingent claim/annuity contract under model uncertainty is determined by G-conditional expectation of the associated discounted payoff. In this paper, by generalizing a nonlinear Feynmac–Kac formula, we show that this sublinear expectation is a viscosity solution of a nonlinear parabolic PDE. This result demonstrates a nice and deep relation between the theory of probability and statistics under uncertainty and second order fully nonlinear parabolic equations.

Keywords: Model uncertainty, G-Brownian motion calculus, Nonlinear Feynman-Kac theorem

Mathematics Subject Classification (2020): 60G65, 60H30

1 Introduction

It is important to measure the uncertain quantities in the real world, such as uncertainties in statistics, measures of risk, and also fully nonlinear super-hedging in finance. A strong mathematical response to such problems is probability theory. However, in some circumstances, probability measure can not be determined in a certain way due to the imperfect knowledge. This situation, where estimators should consider a non-dominated family of probability measures rather than a specific prior, is called *probability model uncertainty* or *ambiguity* and *Knightian uncertainty* in the theory of economics. The book Knight (1921) is a comprehensive discussion about risk and uncertainty. The word "Knightain uncertainty" stems from the name of *Frank Knight* due to his contribution in developing the concept of uncertainty.

The G-expectation theory is one of the frequently used tools to study the attractive and challenging problems including financial pricing and risk management under model uncertainty. In this framework, the notation of sublinear expectation $\hat{\mathbb{E}}$ is indicated which is a monotone, sub additive, constant preserving functional defined on a linear space of random variables.

The pricing of contingent claims in fixed income markets or in insurance modeling under probability uncertainty has been investigated in Holzermann (2021), Holzermann (2022) in the G-setting. Moreover, the authors in Hu and *et al.* (2014) present a relation between

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G-conditional expectation and solutions of (nonlinear) PDEs where the Lipschitz continuity of some functionals is assumed. In the setting of short rate models under a fixed probability measure, a Feynman–Kac formula is proposed in Filipovic (2009) which starting from the PDE associated to the underlying SDE, and proving that its solution can be seen as the conditional expectation of the discounted payoff.

In this paper, inspired by Filipovic (2009), we provide a Feynman–Kac formula under probability uncertainty to include an additional linear term in the associated PDE due to discounting whose coefficients are given in a setting not satisfying the hypothesis of Hu and *et al.* (2014).

1.1 Preliminaries

The G-expectation theory provides very rich and interesting tools to study the model uncertainty. In this section, we breifly discuss the essential preliminaries of this calculus. The reader can refer to Peng (2019) for more details.

Let Ω be a given set and we denote by $\mathcal{H}$ set of random variables which is a vector lattice of real valued functions defined on Ω. Here, $\mathcal{H}$ is a linear space such that $X \in \mathcal{H}$ implies $\varphi(X) \in \mathcal{H}$ for $\varphi \in C_{l,lip}(\mathbb{R})$ where $C_{l,lip}(\mathbb{R})$ denotes the linear space of functions satisfying

$$\varphi(x) - \varphi(y) \leq C_\varphi(1 + |x|^m + |y|^m)$$

for each $x, y \in \mathbb{R}$ and some $C_\varphi > 0$ and $m \in \mathbb{N}$ depending on φ.

Definition 1.1. A sublinear expectation $\hat{\mathbb{E}}$ *is a functional* $\hat{\mathbb{E}}\colon \mathcal{H} \to \mathbb{R}$ *satisfying the following properties*

1. ***Monotonicity:*** *For all* $X, Y \in \mathcal{H}$ *and* $X \leq Y$*, we have* $\hat{\mathbb{E}}[X] \leq \hat{\mathbb{E}}[Y]$*;*
2. ***Constant preserving:*** *For all* $c \in \mathbb{R}$*,* $\hat{\mathbb{E}}[c] = c$*;*
3. ***Sub-additivity:*** *For all* $X, Y \in \mathcal{H}$*, we have* $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$*;*
4. ***Positive homogeneity:*** *For all* $X \in \mathcal{H}$*, we have* $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$ *for all* $\lambda \geq 0$*.*

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ *is called* sublinear expectation space.

Definition 1.2. *A random variable* X *is said to be independent from a random variable* Y *if* $\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(X, y)]y = Y]$*, for* $\varphi \in C_{l,lip}(\mathbb{R} \times \mathbb{R})$*.*

Definition 1.3. *Two random variables* X *and* Y *are identical distributed if* $\hat{\mathbb{E}}[\varphi(X)] = \hat{\mathbb{E}}[\varphi(Y)]$ *for all* $\varphi \in C_{l,lip}(\mathbb{R})$*. In this case, we write* $X \sim Y$ *.*

Definition 1.4. *A random variable* X *in a sublinear expectation* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ *is called* G*-normal distributed if for* $a, b \geq 0$ *it holds*

$$aX + b\bar{X} \sim \sqrt{a^2 + b^2}X,$$

where $a, b \geq 0$ *and* $\bar{X}$ *is an independent copy of* X*. The letter* G *denotes the function* $G : \mathbb{R} \to \mathbb{R}$ *defined by*

$$G(x) := \frac{1}{2}(\overline{\sigma}^2 x^+ - \underline{\sigma}^2 x^-), \quad 0 \leq \underline{\sigma}^2 \leq \overline{\sigma}^2. \tag{1}$$

We denote G*-normal distributed* X *by* $X \sim \mathcal{N}(0, [\underline{\sigma}^2, \overline{\sigma}^2])$ *where* $\overline{\sigma}^2 = \hat{\mathbb{E}}[X^2]$ *and* $\underline{\sigma}^2 = -\hat{\mathbb{E}}[-X^2]$*.*

Definition 1.5. *Let* $G : \mathbb{R} \to \mathbb{R}$ *a given monotonic and sublinear function. A stochastic process* $B = (B_t)_{t \geq 0}$ *on* $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ *is called a* G*-Brownian motion if the following conditions hold:*

1. $B_0 = 0$*;*
2. $B_t \in \mathcal{H}$ *for all* $t \geq 0$*;*

3. For each $t, s \geq 0$, $B_{t+s} - B_t$ is independent of $(B_{t_1}, \ldots, B_{t_n})$ for each $n \in \mathbb{N}$ and $0 \leq t_1 < \ldots < t_n \leq t$. Moreover, $(B_{t+s} - B_t)s^{-\frac{1}{2}}$ is G-normally distributed.

Briefly speaking, a G-Brownian motion is a continuous process with independent and stationary increments under a given sublinear expectation $\hat{\mathbb{E}}$.

Definition 1.6. *Fix now a time horizon $T > 0$ and introduce the space $\Omega_T := C^0([0,T];\mathbb{R})$ of all $\mathbb{R}$-valued continuous paths $\omega = (\omega_t)_{t\in[0,T]}$ with $\omega_0 = 0$. We equip this space with the uniform convergence on compact intervals topology and denote by $\mathcal{F} = \mathcal{B}(\Omega_T)$ the Borel σ-algebra. Moreover, the canonical process $B = (B_t)_{t\in[0,T]}$ on Ω_T is given by $B_t(\omega) := \omega_t$ for $\omega \in \Omega_T$ and $t \in [0,T]$. We define $\Omega_t := \{\omega_{\dot{t}} : \omega \in \Omega_T\}$ and $\mathcal{F}_t := \mathcal{B}(\Omega_t)$ for $t \in [0,T]$.*

Definition 1.7. *Introduce $Lip(\Omega_t)$ as*

$$Lip(\Omega_t) := \{\varphi(B_{t_1}, ..., B_{t_n}) \,|\, n \in \mathbb{N}, t_1, ..., t_n \in [0,t],\ \varphi \in C_{l,lip}(\mathbb{R}^n)\}$$

for $t \in [0,T]$. Then the space $L_G^p(\Omega_t)$ is the completion of $Lip(\Omega_t)$, $t \geq 0$, under the norm $\|\xi\| = (\hat{\mathbb{E}}[|\xi|^p])^{\frac{1}{p}}$ for $p \geq 1$. Next, we introduce the definition of the G-expectation and the G-conditional expectation.

Definition 1.8. *A G-expectation $\hat{\mathbb{E}}$ is a sublinear expectation on $(\Omega, Lip(\Omega_T))$ defined as follows. For $X \in Lip(\Omega_T)$ of the form $X = \varphi(B_{t_1} - B_{t_0}, ..., B_{t_n} - B_{t_{n-1}})$, $0 \leq t_0 < t_1 < ... < t_n \leq T$, $\varphi \in C_{l,lip}(\mathbb{R}^n)$, we set*

$$\hat{\mathbb{E}}[X] := \mathbb{E}[\varphi(\xi_1\sqrt{t_1 - t_0}, ..., \xi_n\sqrt{t_n - t_{n-1}})],$$

where $\xi_1, ..., \xi_n$ are 1-dimensional random variables on a sublinear expectation space $(\tilde{\Omega}, \tilde{\mathcal{H}}, \mathbb{E})$ such that for each $i = 1, ..., n$, ξ_i is G-normally distributed and independent of $(\xi_1, ..., \xi_{i-1})$.

The corresponding canonical process $B = (B_t)_{t\in[0,T]}$ is a G-Brownian motion on the sublinear expectation space $(\Omega_T, Lip(\Omega_T), \hat{\mathbb{E}})$.

Definition 1.9. *Let $X \in Lip(\Omega_T)$ have the representation $X = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_n} - B_{t_{n-1}})$, $\varphi \in C_{l,lip}(\mathbb{R}^n)$, $0 \leq t_1 < ... < t_n \leq T$. Then the G-conditional expectation under $\mathcal{F}_{t_j}$ is defined as*

$$\begin{aligned}\hat{\mathbb{E}}_{t_j}[X] &:= \hat{\mathbb{E}}[X|\mathcal{F}_{t_j}] \\ &= \psi(B_{t_1}, B_{t_2} - B_{t_1}, ..., B_{t_j} - B_{t_{j-1}})\end{aligned}$$

for $j = 1, .., n-1$ where $\psi(x) := \hat{\mathbb{E}}[\varphi(x, B_{t_{j+1}} - B_{t_j}, ..., B_{t_n} - B_{t_{n-1}})]$. The G-expectation and the G-conditional expectation can be extended to sublinear operators $\hat{\mathbb{E}}[.] : L_G^p(\Omega) \to \mathbb{R}$ and $\hat{\mathbb{E}}_t[.] : L_G^p(\Omega_T) \to L_G^p(\Omega_t)$, for $t \in [0,T]$. It is shown that the G-expectation is an upper expectation, i.e., there exists a weakly compact set of probability measures $\mathbb{P}$ such that

$$\hat{\mathbb{E}}[X] = \sup_{P\in\mathbb{P}} E_P[X]$$

for each $X \in L_G^1(\Omega_T)$. As we see that $\hat{\mathbb{E}}$ provides a robust way to measure the risky positions under model uncertainty. Related to this set $\mathbb{P}$, we introduce the Choquet capacity defined by

$$c(A) := \sup_{P\in\mathbb{P}} P(A)$$

for $A \in \mathcal{B}(\Omega_T)$.

Definition 1.10. *For $p \geq 1$, we denote by $M_G^{p,0}(0,T)$ the space of simple integrands. Specifically, for a given partition $\{t_0, ..., t_N\}$ of $[0,T]$, $N \in \mathbb{N}$, we define an element $\eta \in M_G^{p,0}(0,T)$ by*

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) 1_{[t_j, t_{j+1}](t)}$$

where $\xi_i \in L_G^p(\Omega_{t_i})$, $i = 0, ..., N-1$.

Definition 1.11. *For $p \geq 1$, we let $M_G^p(0,T)$ be the completion of $M_G^{p,0}(0,T)$ under the norm*

$$(\int_0^T \hat{\mathbb{E}}[|\eta_t|^p]\, dt)^{\frac{1}{p}}.$$

Note that $M_G^{p,0}(0,T) \subseteq M_G^p(0,T)$. Similar as in the classical Itô's case, the integral with respect to B is first defined for the simple integrands in $M_G^{2,0}$ by the mapping

$$I(\eta) = \int_0^T \eta_s\, dB_s : M_G^{2,0} \to L_G^2(\Omega_T).$$

Then this mapping can be continuously extended to $I : M_G^2 \to L_G^2(\Omega_T)$. As in the classical case the quadratic variation of the G-Brownian motion is defined as the process $\langle B\rangle = (\langle B\rangle_t)_{t\in[0,T]}$ with

$$\langle B\rangle_t = B_t^2 - 2\int_0^t B_s\, dB_s$$

for $t \in [0,T]$. A very interesting new phenomenon of the G-Brownian motion is that its quadratic variation process $\langle B\rangle$ is also a continuous process with independent and stationary increments, and thus can be still regarded as a Brownian motion.

The integral with respect to the quadratic variation

$$\int \eta_s\, d\langle B\rangle_s,$$

is introduced first for $\eta \in M_G^{1,0}(0,T)$ and then extended to $M_G^1(0,T)$.

2 Main Result

In this section, we prove a generalized Feynman–Kac formula in the G-setting, in presence of a linear term in the associated PDE. For this aim, we consider the G-Itô process $X_t^{r,x} = (X_t^{r,x})_{t\in[0,T]}$ given by

$$X_t^{r,x} = x + \int_r^t f(s, X_s^{r,x})\, ds + \int_r^t g(s, X_s^{r,x})\, d\langle B\rangle_s + \int_0^t h(s, X_s^{r,x})\, dB(s), \quad r \leq t \leq T, \quad (2)$$

where $f, g, h : [0,T] \times \mathbb{R} \to \mathbb{R}$ are deterministic functions such that $f(t,.), g(t,.), h(t,.)$ are Lipschitz-continuous functions for every $t \in [0,T]$ and $f(.,x), g(.,x), h(.,x)$ are continuous in t for every $x \in \mathbb{R}$. We aim to examine the G-conditional expectation follows as

$$\hat{\mathbb{E}}[\varphi(T, X_T) e^{-\int_r^T X_s\, ds} | X_r = x] \quad (3)$$

for any $(r,x) \in [0,T] \times (0,\infty)$ where φ is a bounded real-valued continuous function. Note that φ represents the payoff of an interest rate derivative or of a life insurance liability in applications.

An analogous of a Feynman–Kac formula within the framework of G-expectation is given in Hu and *et al.* (2014). This result connects the solution of a G-BSDE to the unique viscosity solution of a fully nonlinear PDE, whose nonlinearity is given by a term that depends on the function G representing the uncertainty about the volatility. The authors are able to deal with viscosity (i.e., non necessarily classical) solutions to the associated PDEs assuming the Lipschitz continuity of the coefficients of the G-BSDE. However, due to the presence of the discounting term $e^{-\int_r^T X_s\, ds}$ in (3), these conditions are not satisfied in our framework, so that we cannot apply this formula directly. Moreover, Hu and *et al.* (2014) start from a G-conditional expectation and prove that this solves a fully nonlinear PDE.

In the sequel, we aim to extend the approach in Filipovic (2009), which starts from PDE, to the G-setting. In particular, we assume that X is quasi-surely strictly positive and solves the given G-SDE (2) and we establish a relation between a nonlinear PDE and G-conditional expectation of a discounted payoff.

2.1 Feynman-Kac formula for a bounded payoff

In this section, we provide a Feynman–Kac formula under volatility uncertainty to include an additional linear term in the associated PDE due to discounting. For this aim, we present a theorem related to the bounded payoffs.

Assumption 2.1. *The function φ belongs to $C([0,T]\times\mathbb{R}^+)$ and is bounded by a constant $M_0>0$.*

Theorem 2.2. *Let $\varphi\in C([0,T]\times\mathbb{R}^+)$ satisfy Assumption 2.1. Given $(r,x)\in[0,T]\times(0,\infty)$, assume that the process $X^{r,x}=(X_t^{r,x})_{t\in[r,T]}$ in (2) is quasi-surely strictly positive. Define the function $u:[0,T]\times(0,\infty)\to\mathbb{R}$ by*

$$u(r,x):=\hat{\mathbb{E}}_r[\varphi(T,X_T^{r,x})e^{-\int_r^T X_s^{r,x}\,ds}].$$

Then u is a viscosity solution of the PDE

$$\begin{aligned} &u_t+2G(u_x g(t,x)+\frac{1}{2}u_{xx}h^2(t,x))+f(t,x)u_x-xu=0,\quad (t,x)\in Q,\\ &u=\varphi,\quad (t,x)\in\partial' Q \end{aligned} \tag{4}$$

where G is defined by (1), $Q=(0,T)\times(0,\infty)$ and $\partial' Q=\{T\}\times\mathbb{R}^+\cup(0,T)\times\{0\}$.

2.1.1 The sketch of the proof

In order to perform the analysis illustrated above, we have to use both stochastic calculus in the G-setting and the theory of nonlinear PDEs. In general the PDE (4) does not admit a classical $C^{1,2}$ solution. That is because, we substitute the coefficients of the original PDE with some bounded and $C^{1,2}$ cut-off functions depending on a parameter $\varepsilon\in(0,1)$. In this way, we are able to show that the approximated PDE admits a viscosity solution u_ε defined on an unbounded domain, which is additionally $C^{1,2}$ inside a bounded domain D_ε depending on ε by following Krylov (1987).

For this aim, for fixed $\varepsilon\in(0,1)$, we define the intervals $D_\varepsilon:=(\varepsilon,\varepsilon^{-1})$ and the random time

$$\tau_\varepsilon^{r,x}:=\inf\{s\in[r,T]:X_s^{r,x}\in[\varepsilon,\varepsilon^{-1}]\}\wedge T,$$

where we use the convention $\inf\varnothing:=+\infty$. Then we can apply the G-Itô's formula to prove that

$$u_\varepsilon(r,x)=\hat{\mathbb{E}}_r[u_\varepsilon(T\wedge\tau_\varepsilon^{r,x},X_{T\wedge\tau_\varepsilon^{r,x}}^{r,x})e^{-\int_r^{T\wedge\tau_\varepsilon^{r,x}}(X_s^{r,x}+\varepsilon)\,ds}],$$

is the viscosity solution of the regularized PDE corresponding to the PDE (4) based on this fact that $u_\varepsilon\in C^{1,2}$. The reader can find more details in Akhtari and *et al.* (2023). Finally, we prove that

$$\lim_{\varepsilon\to0}|u_\varepsilon(r,x)-u(r,x)|=0$$

for any $(r,x)\in[0,T]\times(0,\infty)$ which finishes the proof.

Discussion and Results

We proved that the G-conditional expectation of a discounted payoff is a limit of the viscosity solutions of a family of regularized PDEs which is also $C^{1,2}$ on a bounded domain depending on the regularizing parameter with terminal condition φ at time $T>0$ when the regularity parameter goes to zero. Indeed, such a limit is itself a viscosity solution of the limit PDE.

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