پانزدهمین سمینار احتمال
و فرآیندهای تصادفی
۸ و ۹ شهریور ۱۴۰۴
دانشگاه کردستان

15th Seminar On Probability and Stochastic Processes
August 30-31, 2025
University of Kurdistan

انجمن آمار ایران

On the stability and convergence of a finite difference scheme for stochastic partial differential equations

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Abstract

In this paper, we introduce a stochastic finite difference scheme for the numerical solution of the Itô stochastic advection–diffusion equation with a one–dimensional white noise process. The main concepts of deterministic difference schemes namely, consistency, stability, and convergence are theoretically established for the stochastic case. Finally, a numerical example is provided to demonstrate the efficiency of the proposed stochastic difference.

Keywords: Stochastic partial differential equations, consistency, stability, convergence.
Mathematics Subject Classification (2020): 60H15, 60H35, 65M12.

1 Introduction

Many physical applications and engineering systems that are typically modeled using deterministic partial differential equations (PDEs) exhibit uncertainties due to various stochastic perturbations. As a result, to provide a more precise representation of the behavior of such phenomena, they are often better modeled using stochastic partial differential equations (SPDEs). The study of SPDEs has become a significant area of research in both applied and pure mathematics, as well as fluid mechanics, geophysics, and theoretical physics. It is important to note that obtaining exact solutions for SPDEs is often challenging, making it essential to investigate their numerical solutions using effective numerical methods. Several numerical techniques are widely used to approximate solutions to SPDEs. These include stochastic finite difference (SFD) scheme, finite element methods (EEMs) [9], and spectral methods. In [1, 2, 4, 6, 7], Namjoo et al. investigated the stability and convergence of several SFD schemes for SPDEs. In this paper, we develop a finite difference scheme for the stochastic advection–diffusion equation with one–dimensional Wiener process. The structure of the paper is organized as follows: Section 2 introduces an explicit SFD scheme for numerical solutions of SPDEs. Section 3 analyzes the consistency, stability, and convergence of the proposed SFD scheme. Numerical simulations are presented in Section 4 to confirm the theoretical finding.

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2 An explicit finite difference scheme for SPDEs

In this section, we concentrate on developing an efficient numerical method for approximating the solution of the stochastic advection–diffusion equation, as outlined in [6].

$$u_t(x,t) + \nu u_x(x,t) = \gamma u_{xx}(x,t) + \sigma u(x,t)\dot{W}(t), \qquad x \in [0,1], \qquad t \in [0,1], \tag{1}$$

with initial and boundary conditions

$$u(x,0) = u_0(x), \qquad u(0,t) = u_1(t), \qquad u(1,t) = u_2(t), \qquad (x,t) \in [0,1] \times [0,1].$$

Here, ν and γ are positive constants, and $W(t)$ denotes a one–dimensional standard Wiener process, where the white noise $\dot{W}(t)$ has a Gaussian distribution with zero mean [3]. Consider a uniform grid in both space and time with stepsizes Δx and Δt, respectively. Let u_k^n denote the approximate solution at the spatial point $x_k = k\Delta x$ and time $t_n = n\Delta t$, where $\Delta x = x_{k+1} - x_k$ and $\Delta t = t_{n+1} - t_n$, for $0 \le k \le M-1$ and $0 \le n \le N-1$. To develop an SFD scheme for the SPDE (1), the partial derivatives in time and space can be approximated using methods described in [8].

$$\begin{aligned} u_t(k\Delta x, n\Delta t) &\approx \frac{u_k^{n+1} - u_k^n}{\Delta t}, \\ u_x(k\Delta x, n\Delta t) &\approx \frac{u_{k+1}^n - u_{k-1}^n}{2\Delta x}, \\ u_{xx}(k\Delta x, n\Delta t) &\approx \frac{u_{k-1}^n - 2u_k^n + u_{k+1}^n}{\Delta x^2}. \end{aligned} \tag{2}$$

In fact, (2) represents a second order approximation of $u_{xx}(x_k, t_n)$, with the truncation error being order $O(\Delta x^2)$. Therefore, the scheme provides an approximation for the stochastic advection–diffusion equation as follows:

$$u_k^{n+1} = (1 - 2\gamma\Theta)u_k^n + \left(\gamma\Theta - \frac{\nu\lambda}{2}\right) u_{k+1}^n + \left(\gamma\Theta + \frac{\nu\lambda}{2}\right) u_{k-1}^n + \sigma u_k^n \Delta W_n, \tag{3}$$

where $\lambda = \frac{\Delta t}{\Delta x}$, $\Theta = \frac{\Delta t}{\Delta x^2}$. The term $\Delta W_n = W((n+1)\Delta t) - W(n\Delta t)$ denotes the increment of the Wiener process, where $\Delta W_n \sim \mathcal{N}(0, \Delta t)$.

3 Qualitative properties of the proposed scheme

In this section, we verify that the stochastic difference scheme (3) is consistent, stable, and convergent. To do this, consider a stochastic partial differential equation of the form $\mathcal{L}\mathcal{U} = \mathcal{F}$. Let $\mathcal{L}_k^n u_k^n = \mathcal{F}_k^n$ denote the discretized scheme of this equation using the proposed finite difference scheme. To analyze consistency, stability, and convergence, we define a norm for the numerical solutions. Let $\{u_k^n\}$ stand for the sequence of approximations generated by the stochastic difference scheme (3). For this sequence, we define $||u^n|| = (\max\limits_{0 \le m \le M} |u_m^n|^2)^{\frac{1}{2}}$, where $u^n = (u_0^n, u_1^n, ..., u_M^n)$. More details regarding consistency, stability, and convergence can be found in [5].

Definition 3.1. *The stochastic finite difference scheme $\mathcal{L}_k^n u_k^n = \mathcal{F}_k^n$ is said to be pointwise consistent with the SPDE $\mathcal{L}U = F$ at the point (x,t) if for any continuously differentiable function $\Upsilon = \Upsilon(x,t)$ in the mean square sense, the following holds:*

$$\mathbb{E}\left|(\mathcal{L}\Upsilon - F)\big|_{x_k}^{t_n} - (\mathcal{L}_k^n \Upsilon(x_k, t_n) - \mathcal{F}_k^n)\right|^2 \to 0,$$

as $\Delta x \to 0$, $\Delta t \to 0$, and $(k\Delta x, (n+1)\Delta t) \to (x,t)$.

Theorem 3.2. *The stochastic difference scheme (3) is mean square consistent, as defined in 3.1.*

Proof. Let $\Upsilon(x,t)$ be a smooth function. Then, one can obtain

$$\begin{aligned}\mathcal{L}(\Upsilon(x,t))|_{x_k}^{t_n} &= \Upsilon(k\Delta x,(n+1)\Delta t)-\Upsilon(k\Delta x,n\Delta t)\\ &+v\int_{n\Delta t}^{(n+1)\Delta t}\Upsilon_x(k\Delta x,s)ds-\gamma\int_{n\Delta t}^{(n+1)\Delta t}\Upsilon_{xx}(k\Delta x,s)ds\\ &-\sigma\int_{n\Delta t}^{(n+1)\Delta t}\Upsilon(k\Delta x,s)dW(s),\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_k^n(\Upsilon) &= \Upsilon(k\Delta x,(n+1)\Delta t)-\Upsilon(k\Delta x,n\Delta t)\\ &+v\frac{\Delta t}{2\Delta x}[\Upsilon((k+1)\Delta x,n\Delta t)-\Upsilon((k-1)\Delta x,n\Delta t)]\\ &-\gamma\frac{\Delta t}{\Delta x^2}[\Upsilon((k+1)\Delta x,n\Delta t)-2\Upsilon(k\Delta x,n\Delta t)+\Upsilon((k-1)\Delta x,n\Delta t)]\\ &-\sigma\Upsilon(k\Delta x,n\Delta t)\Delta W_n.\end{aligned}$$

Using the property of Itô integrals, the difference's mean square expectation satisfies:

$$\begin{aligned}&\mathbb{E}|\mathcal{L}(\Upsilon)|_k^n-\mathcal{L}_k^n(\Upsilon)|^2=\mathbb{E}\Big|\nu\int_{n\Delta t}^{(n+1)\Delta t}\Big(\Upsilon_x(k\Delta x,s)-\frac{\Upsilon((k+1)\Delta x,n\Delta t)-\Upsilon((k-1)\Delta x,n\Delta t)}{2\Delta x}\Big)ds\\ &-\gamma\int_{n\Delta t}^{(n+1)\Delta t}\Big(\Upsilon_{xx}(k\Delta x,s)-\frac{1}{(\Delta x)^2}\big[\Upsilon((k-1)\Delta x,n\Delta t)-2\Upsilon(k\Delta x,n\Delta t)+\Upsilon((k+1)\Delta x,n\Delta t)\big]\Big)ds\\ &-\sigma\int_{n\Delta t}^{(n+1)\Delta t}\big(\Upsilon(k\Delta x,s)-\Upsilon(k\Delta x,n\Delta t)\big)\,dW(s)\Big|^2\\ &\le 4\nu^2\mathbb{E}\Big|\int_{n\Delta t}^{(n+1)\Delta t}\Big(\Upsilon_x(k\Delta x,s)-\frac{\Upsilon((k+1)\Delta x,n\Delta t)-\Upsilon((k-1)\Delta x,n\Delta t)}{2\Delta x}\Big)ds\Big|^2\\ &+4\gamma^2\mathbb{E}\Big|\int_{n\Delta t}^{(n+1)\Delta t}\Big(\Upsilon_{xx}(k\Delta x,s)-\frac{1}{(\Delta x)^2}\big[\Upsilon((k-1)\Delta x,n\Delta t)\\ &-2\Upsilon(k\Delta x,n\Delta t)+\Upsilon((k+1)\Delta x,n\Delta t)\big]\Big)\,ds\Big|^2\\ &+4\sigma^2\int_{n\Delta t}^{(n+1)\Delta t}\mathbb{E}\,|\Upsilon(k\Delta x,s)-\Upsilon(k\Delta x,n\Delta t)|^2\,ds.\end{aligned}$$

Since $\Upsilon(x,t)$ is deterministic, it implies that

$$\mathbb{E}|\mathcal{L}(\Upsilon)|_k^n-\mathcal{L}_k^n(\Upsilon)|^2\to 0,\quad \text{as } n,k\to\infty.$$

Definition 3.3. *A stochastic difference scheme is stable in the mean square sense with respect to a norm if there exist positive constants $\overline{\Delta x}$ and $\overline{\Delta t}$, along with non–negative constants κ and α, such that*

$$\mathbb{E}\|u^{n+1}\|^2\le \kappa e^{\alpha t}\mathbb{E}\|u^0\|^2,$$

for all time step $t=(n+1)\Delta t$, where $0<\Delta x\le\overline{\Delta x}$, and $0<\Delta t\le\overline{\Delta t}$.

Theorem 3.4. *The stochastic difference scheme (3), under the conditions $\nu\lambda^2\le\gamma\Theta\le\frac{1}{2}$, for $t=(n+1)\Delta t$, is stable with respect to the introduced norm.*

Proof. Applying the $\mathbb{E}|\cdot|^2$ for the stochastic difference scheme (3) and utilizing the independence of Wiener process increments, one obtains that

$$\begin{aligned}\mathbb{E}|u_k^{n+1}|^2 =&\mathbb{E}\left|(1-2\gamma\Theta)u_k^n+\left(\gamma\Theta-\frac{\nu\lambda}{2}\right)u_{k+1}^n+\left(\gamma\Theta+\frac{\nu\lambda}{2}\right)u_{k-1}^n+\sigma u_k^n\Delta W_n\right|^2\\ =&\mathbb{E}\left|(1-2\gamma\Theta)u_k^n+\left(\gamma\Theta-\frac{\nu\lambda}{2}\right)u_{k+1}^n+\left(\gamma\Theta+\frac{\nu\lambda}{2}\right)u_{k-1}^n\right|^2\\ &+2\mathbb{E}\left[\left((1-2\gamma\Theta)u_k^n+\left(\gamma\Theta-\frac{\nu\lambda}{2}\right)u_{k+1}^n+\left(\gamma\Theta+\frac{\nu\lambda}{2}\right)u_{k-1}^n\right)(\sigma u_k^n\Delta W_n)\right]\\ &+\sigma^2\Delta t\mathbb{E}|u_k^n|^2\end{aligned}$$

$$= \mathbb{E}\left|(1-2\gamma\Theta)u_k^n + (\gamma\Theta - \frac{\nu\lambda}{2})u_{k+1}^n + (\gamma\Theta + \frac{\nu\lambda}{2})u_{k-1}^n\right|^2 + +\sigma^2\Delta t\mathbb{E}\left|u_k^n\right|^2.$$

Using the inequalities $\nu\lambda^2 \leq \gamma\Theta \leq \frac{1}{2}$, one concludes that

$$\begin{aligned}
\mathbb{E}|u_k^{n+1}|^2 &\leq (1-2\gamma\Theta)^2\mathbb{E}|u_k^n|^2 + (\gamma\Theta - \frac{\nu\lambda}{2})^2\mathbb{E}|u_{k+1}^n|^2 \\
&+ (\gamma\Theta + \frac{\nu\lambda}{2})^2\mathbb{E}|u_{k-1}^n|^2 \\
&+ 2(1-2\gamma\Theta)(\gamma\Theta - \frac{\nu\lambda}{2})\mathbb{E}(|u_k^n||u_{k+1}^n|) \\
&+ 2(1-2\gamma\Theta)(\gamma\Theta + \frac{\nu\lambda}{2})\mathbb{E}(|u_k^n||u_{k-1}^n|) \\
&+ 2(\gamma\Theta - \frac{\nu\lambda}{2})(\gamma\Theta + \frac{\nu\lambda}{2})\mathbb{E}(|u_{k+1}^n||u_{k-1}^n|) \\
&+ \sigma^2\Delta t\mathbb{E}|u_k^n|^2 \\
&\leq (1-2\gamma\Theta)^2\mathbb{E}|u_k^n|^2 + (\gamma\Theta - \frac{\nu\lambda}{2})^2\mathbb{E}|u_{k+1}^n|^2 \\
&+ (\gamma\Theta + \frac{\nu\lambda}{2})^2\mathbb{E}|u_{k-1}^n|^2 \\
&+ (1-2\gamma\Theta)(\gamma\Theta - \frac{\nu\lambda}{2})\{\mathbb{E}|u_k^n|^2 + \mathbb{E}|u_{k+1}^n|^2\} \\
&+ (1-2\gamma\Theta)(\gamma\Theta + \frac{\lambda h}{2})\{\mathbb{E}|u_k^n|^2 + \mathbb{E}|u_{k-1}^n|^2\} \\
&+ (\gamma\Theta - \frac{\nu\lambda}{2})(\lambda\Theta + \frac{\nu\lambda}{2})\{\mathbb{E}|u_{k+1}^n|^2 + \mathbb{E}|u_{k-1}^n|^2\} + \sigma^2\Delta t\,\mathbb{E}|u_k^n|^2 \\
&\leq (1-2\gamma\Theta)^2\sup_k\mathbb{E}|u_k^n|^2 + (\gamma\Theta - \frac{\nu\lambda}{2})^2\sup_k\mathbb{E}|u_k^n|^2 \\
&+ (\gamma\Theta + \frac{\nu\lambda}{2})^2\sup_k\mathbb{E}|u_k^n|^2 + 2(1-2\gamma\Theta)(\gamma\Theta - \frac{\nu\lambda}{2})\sup_k\mathbb{E}|u_k^n|^2 \\
&+ 2(1-2\gamma\Theta)(\gamma\Theta + \frac{\nu\lambda}{2})\sup_k\mathbb{E}|u_k^n|^2 + 2(\gamma\Theta - \frac{\nu\lambda}{2})(\gamma\Theta + \frac{\nu\lambda}{2})\sup_k\mathbb{E}|u_k^n|^2 \\
&+ \sigma^2\Delta t\sup_k\mathbb{E}|u_k^n|^2 \\
&= \left((1-2\gamma\Theta) + (\gamma\Theta - \frac{\nu\lambda}{2}) + (\gamma\Theta + \frac{\nu\lambda}{2})\right)^2 + \sigma^2\Delta t\Big)\sup_k\mathbb{E}|u_k^n|^2 \\
&= (1+\sigma^2\Delta t)\sup_k\mathbb{E}|u_k^n|^2.
\end{aligned}$$

Since this holds for all k, it follows that

$$\sup_k\mathbb{E}|u_k^{n+1}|^2 \leq (1+\sigma^2\Delta t)\sup_k\mathbb{E}|u_k^n|^2.$$

Note that $\Delta t = \frac{t}{(n+1)}$, which implies that

$$\begin{aligned}
\mathbb{E}||u^{n+1}||^2 &\leq (1+\sigma^2\Delta t)\mathbb{E}||u^n||^2 \\
&\vdots \\
&\leq (1+\sigma^2\Delta t)^{n+1}\mathbb{E}||u^0||^2 \\
&\leq e^{\sigma^2(n+1)\Delta t}\mathbb{E}||u^0||^2 \\
&= e^{\sigma^2 t}\mathbb{E}||u^0||^2.
\end{aligned}$$

By Setting $\kappa = 1$ and $\alpha = \sigma^2$, we deduce that the stochastic difference scheme (3) is stable. □

Definition 3.5. *A stochastic difference scheme $\mathcal{L}_k^n u_k^n = \mathcal{F}_k^n$ that approximates the SPDE $\mathcal{L}\mathcal{U} = \mathcal{F}$ is said to be mean square convergent at time $t = (n+1)\Delta t$, if $\mathbb{E}|u^{n+1} - \mathcal{U}^{n+1}|^2 \to 0$, as $\Delta x \to 0$, $\Delta t \to 0$.*

Theorem 3.6. *The stochastic difference scheme (3) for $\frac{\nu\lambda}{2} \leq \gamma\Theta \leq \frac{1}{2}$ is convergent in mean square sense with repsect to the defined norm.*

Proof. According to the stochastic Lax-Richtmyer theorem, the stochastic difference scheme (3) is convergent in the mean square sense with respect to the specified norm. □

4 Numerical implementations and results

In this section, we present the numerical results of the stochastic difference scheme (3). All numerical results were obtained using MATLAB 2024.

Example 1. We analyze the performance of the proposed stochastic scheme for the SPDE:

$$u_t(x,t) = 0.01u_{xx}(x,t) + 2u(x,t)\dot{W}(t), \qquad (x,t) \in [0,1] \times [0,1], \tag{4}$$

with the initial and boundary conditions

$$\begin{aligned} u(x,0) &= \exp(-100(x-0.2)^2), && x \in [0,1], \\ u(0,t) &= \frac{1}{\sqrt{4t+1}} \exp\left(-\frac{4}{4t+1}\right), \\ u(1,t) &= \frac{1}{\sqrt{4t+1}} \exp\left(-\frac{64}{4t+1}\right), && t \in [0,1]. \end{aligned}$$

In the absence of the stochastic term, the exact solution to problem (4) is given by

$$u(x,t) = \frac{1}{\sqrt{4t+1}} \exp\left(\frac{-100(x-0.2)^2}{4t+1}\right).$$

The stochastic difference scheme is given by

$$u_k^{n+1} = u_k^n + \frac{\Theta}{100}(u_{k-1}^n - 2u_k^n + u_{k+1}^n) + \sigma u_k^n \big(W((n+1)\Delta t) - W(n\Delta t)\big), \tag{5}$$

where $\Delta x = \frac{1}{M}$, and $\Delta t = \frac{1}{N}$ for some positive integers M and N. This scheme is conditionally stable under the constraints $\frac{\nu\lambda}{2} \leq \gamma\Theta \leq \frac{1}{2}$. Consequently, if $M = 100$, the stability (or convergence) condition requires $\Delta t \leq 0.005$ or equivalently $N \geq 200$. This is illustrated in Figure 1. It is important to note that the scheme becomes unstable when $N = 100$. According to Table 1. the results of $\mathbb{E}(u(0.2,1))$ are distroyed when $N < 200$, whereas stable results are achieved for $N \geq 200$.

N	$\mathbb{E}(u(0.2,1))$
100	$5.4762E+39$
110	$2.3617E+38$
120	$3.0925E+36$
150	$-4.7346E+25$
200	0.3414
205	0.3447
210	0.3217
300	0.3271

Table 1: Solve the SPDE (4) using the proposed stochastic difference scheme.

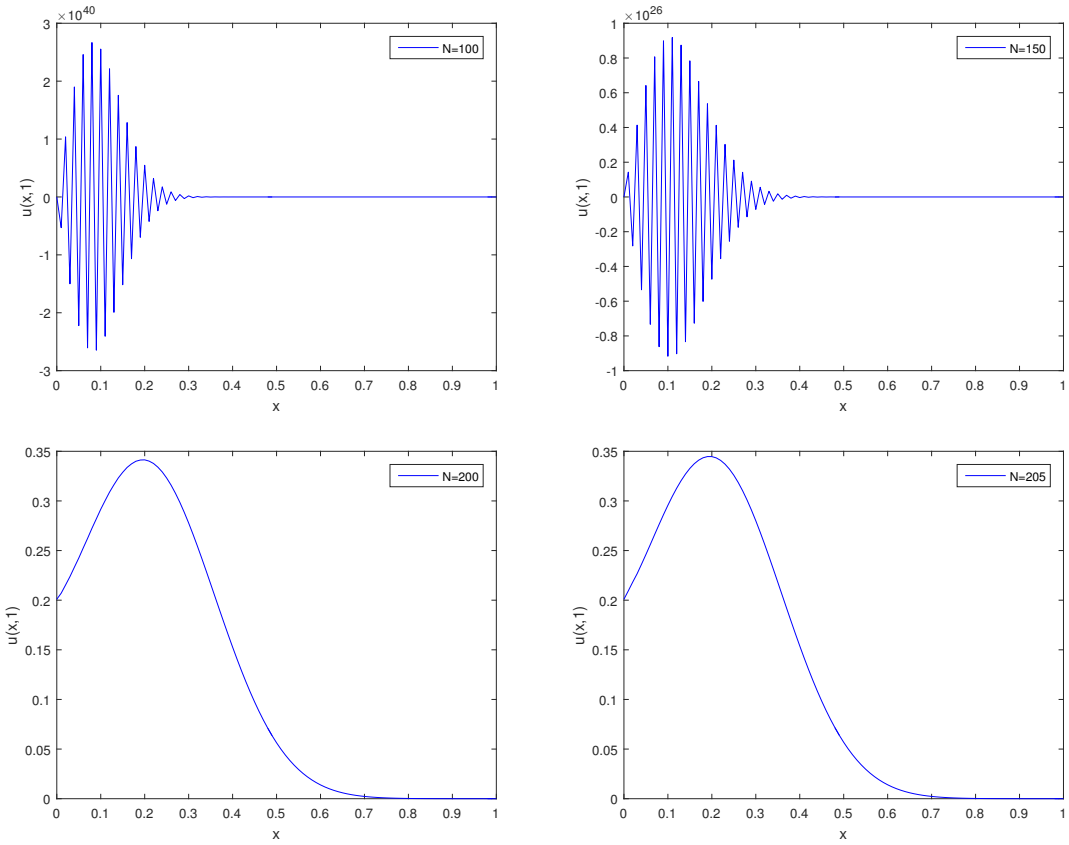


Figure 1: Analysis of the stability of the stochastic difference scheme (5) for different values of N.

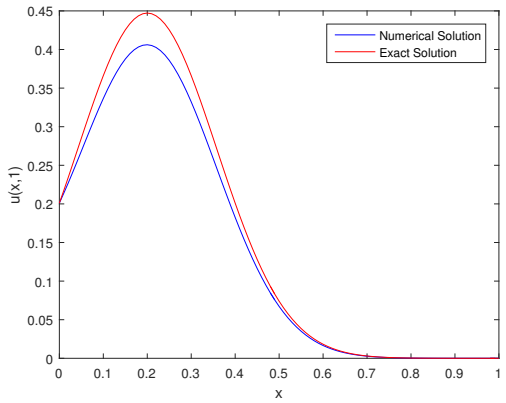


Figure 2: Comparison between the deterministic and stochastic numerical solutions of SPDE (4) using the stochastic difference scheme (5).

5 Conclusion and future work

In this paper, we have presented an explicit stochastic finite difference scheme for a class of SPDEs. We have also analyzed the scheme's consistency, stability, and convergence. The results from our numerical simulations indicate that the proposed scheme is effective for a

variety of SPDEs. Our future research will focus on developing nonstandard stochastic finite difference schemes for these equations.

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