پانزدهمین سمینار احتمال
و فرآیندهای تصادفی
۸ و ۹ شهریور ۱۴۰۴
دانشگاه کردستان

15th Seminar On Probability and Stochastic Processes
August 30-31, 2025
University of Kurdistan

انجمن آمار ایران

Allocated Optimal Strategies to the Wealth Process of a Diffusion Risk Model

Abouzar Bazyari*

Department of Statistics, Faculty of Intelligent Systems Engineering and Data Science,

Persian Gulf University, Bushehr, Iran

Abstract

In this paper, we obtain a closed-form solution for a diffusion risk model which optimally allocates his/her wealth among the following coffecients: a risky asset, a free bank account, and a stock. The following relationships between the optimal amount invested in the security and the risk premium are obtained: (i) for a risk premium greater than one, the investor will optimally invest a positive amount in the bond, and (ii) for a risk premium equal to one, the insurance company will optimally invest nothing in the bond. Although the inclusion of a credit-related financial product in the portfolio selection if risk models is more realistic, no closed-form solutions to date are given in the literature when a recovery value is considered in the event of a default. We solve the optimal portfolio problem of insurance company for the representative investor with a specified utility function. Moreover, the implications of the analytic result for asset allocation are discussed and carried out sensitivity analysis by adopting benchmark parameter values.

Keywords: Itô's formula, Optimal Strategy, Portfolio optimization, Defaultable security, Credit risk, Recovery of market value.
Mathematics Subject Classification (2020): 62G32, 62F99, 62E20.

1 Introduction

Insurance is a means of protection from random and untoward events, which can lead to significant financial losses. It is a method of risk management, that is used in order to hedge against a contingent loss risk. Risk theory came into being and was developed in order to provide a basis for the existence and significance of the insurance system. One of the most fundamental problems in risk theory is the ruin problem. It analyzes the behavior of a stochastic process, that represents the evolution of the capital of an insurance company. According to Dickson (2005) this model is a simplification of reality, however, this is a useful model, which can give some insight into the characteristics of an insurance operation. In the financial markets, high corporate bonds are increasingly attractive to investors. Compared to stocks and free bonds,

*Corresponding author, ab_bazyari@pgu.ac.ir

corporate bonds provide different risk-return profiles to investors and are often sought after because of their high yields. Although rare, corporate bond defaults do occur. One of the largest defaults in the history of a defaultable security was the default by Enron. Kmart also filed a bankruptcy, which led to significant losses to investors. As a result, optimal portfolio problems with defaultable securities have become an important area of research. In spite of several major contributions to the theory of optimal portfolio selection only a handful of papers (e.g. Korn and Kraft (2003), Walder (2002), Hou and Jin (2002), and Hou (2003)) considered the case where one or more of the securities in the portfolio are subject to default. Walder (2002) considered the optimal portfolio problem by assuming that the investor can invest in a treasury bond and a portfolio of corporate zero-coupon bonds. Hou and Jin (2002) and Hou (2003) address the optimal portfolio problem of the investor by giving the investor the ability to allocate her wealth among a stock, a default-free bank account, and a credit-risky financial instrument.

Aydoğan and Steffensen (2024) determined the optimal investment strategies for the agent under the specific situations arising from the pure-diffusion and pure-jump market cases. Lacker and Zariphopoulou (2019) obtained the optimal investment strategies for n agents with relative performance concerns under CARA and CRRA utilities. Bäuerle and Göll (2024) have recently extended their studies by analyzing the effect of the price impact on equilibrium strategies. Deng et al. (2024) introduced another extension of Lacker and Zariphopoulou (2019) in which they investigate a mean-field game strategy in a partially observable market. Some other studies derived the optimal strategies assuming stochastic volatility, for instance, Zhu et al. (2019), Kraft (2020) and Zariphopoulou (2024).

In the present paper, we derive the optimal amount of wealth allocated to a defaultable security as well as to a stock and to a default-free money market account. The following relationships between the optimal amount invested in the defaultable security and the default-risk premium are obtained: (i) for a default-risk premium greater than one, the investor will optimally invest a positive amount in the defaultable bond, and (ii) for a risk premium equal to one, the investor will optimally invest nothing in the defaultable bond. A risk premium greater than one can be interpreted as the market fairly pricing the default risk in the defaultable bond. On the other hand, a default-risk premium equal to one, can be interpreted as the market not pricing the default risk in the defaultable bond.

The rest of the paper is organized as follows: Section 2 presents the dynamics of a defaultable bond, a stock as well as a savings account under two equivalent probability measures Q and P which will be used for the portfolio optimization in Section 3. Implications of the analytic result for asset allocation are discussed in Section 4. Section 5 carries out sensitivity analysis by adopting benchmark parameter values from the literature. Section 6 summarizes the results and concludes the paper.

2 Dynamics of Financial Securities

In the sequel to obtain our goal, we we will work on the probability space $(\Omega, \mathcal{F}, P)$, which is endowed with the information filtration $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0}$ which carries all stochastic quantities and right continuity and it is often called an enlarged filtration given by $\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t$. In the probability space, P denotes the martingale probability measure. The filteration $\mathcal{G}_t$ is assumed to be generated by the Winner process which shows the change of stock price and the filteration $\mathcal{H}_t$ is shows a Poisson process that is used to denote the arrival of risks. In the probability space Q denotes the martingale probability measure (or risk neutral measure), which is assumed to be equivalent to the some real-world probability measure which we denote by P.

Now we are ready to define a default process.

Definition 2.1. *A nondecreasing right continuous process which makes discrete jumps at a*

random time τ is called default process. We denote a default process by $H_t = 1_{\tau \le t}$, where 1 represents the indicator function which takes the value of one if there is a jump and zero otherwise and $H_u^- = \lim_{s \to u} H_s$.

The default process H is assumed to be a Poisson process with a constant intensity h.

Definition 2.2. *The default process $H = (H_t)_{t \ge 0}$ is said to be a Poisson process if*
(i)$H_0 = 0$.
(ii) $H = (H_t)_{t \ge 0}$ has independent increments.
(iii) The number of events in any interval of length t has a Poisson distribution with mean ht. That is, for all $s, t \ge 0$.

$$P\big(H_{t+s} - H_s = n\big) = e^{-ht} \frac{(ht)^n}{n!}, \qquad n \ge 0.$$

Here, h represents the intensity of the Poisson process, which measures the arrival rate of a default. The mean arrival rate ht is called the compensator for the Poisson process. We now define a martingale (compensated) Poisson process.

Definition 2.3. *The martingale default process is thus given by the following equation:*

$$M_t = H_t - \int_0^t (1 - H_{u^-}) h du.$$

The stochastic differential equation (SDE) of compensated Poisson process defined above is $dM_t = dH_t - h(1 - H_{t^-})dt$. We extensively utilize this stochastic differential equation (SDE) to derive the dynamics of the defaultable bond.

2.1 Dynamics of bond price process and stock price under Q

We assume that there exists a defaultable zero coupon bond with a maturity date $T1$. The value of the bond after default is assumed to be zero. In the case of default the investor recovers a fraction of the market value of the defaultable bond just prior to default. The loss rate is denoted by ζ which is assumed to be constant and take value between zero and one. The arrival rate of the default h^Q of the Poisson process is also assumed to be constant. Table 1 summarizes the parameter's values.

Table 1: Parameter Definition and values

Symbol	*Denition*
h^Q	risk neutral default intensity
$\frac{h^Q}{h^p}$	default risk premium
γ	risk aversion parameter
ζ	write-down rate
μ	instantaneous rate of return of stock
σ	instantaneous volatility of stock's return
r	risk free short rate of interest

We will now define the price process for a bond under the assumption of constant interest rate and intensity under the risk neutral measure Q as follows:

$$p(t, T_1) = 1_{\tau < t} e^{-(r+\delta)(T_1 - t)} + 1_{\tau \le t}(1 - \zeta) e^{-(r+\delta)(T_1 - t)} e^{(t-\tau)}. \tag{1}$$

In the equation (1), $p(t, T_1)$ being a fictitious security because it is not really a traded security. It actually consists of two components, which are mutually exclusive, i.e., the component of the left hand side is the actual defaultable bond price given that default has yet to occur,

whereas the component of the right hand side represents the value (or recovered amount) of an already defaulted bond. By accounting for the second component we are implicitly modeling the difference between the bond price if there is a full recovery and fractional recovery at default. The explicit inclusion of the recovery is the first innovation of this paper and the measure probability Q dynamics of the price process of the defaultable bond is derived in the following lemma.

Lemma 2.4. *The Q dynamics for the price process of the defaultable bond at time t is represented in terms of the compensated jump process M^Q, which is Q martingale process, such as:*

$$dp(t,T_1) = rp(t,T_1)dt - \zeta e^{-(r+\delta)(T_1-t)}dM^Q. \tag{2}$$

Proof. Applying the Itô's formula on equation (1). □

Now we consider that the agent can invest money in the bank at the constant short rate of interest r, i.e. he/she has access to the risk-free asset B_t with

$$dB_t = rB_t dt. \tag{3}$$

The investor can also invest in a risky stock with price process S_t, the dynamics under Q is given by the following diffusion equation:

$$dS_t = S_t(rdt + \sigma dW_t^Q), \tag{4}$$

where σ instantaneous volatility of the equity return process is assumed to be constant. Note that the expected return under Q for each security is the risk free interest rate.
We state the theorem without proof.

Theorem 2.5. *For positive interest rate r, the dynamics of bond price is represented as the exponential form of SDE as follows:*

$$\begin{aligned} dp(t,T_1) &= (1-H_t)(r+\delta-\delta\Delta)p(t,T_1)dt + H_t rp(t,T_1)dt \\ &\quad -(1-H_{t^-})\zeta p(t,T_1)dM_t^Q \\ &= p(t^-,T_1)\big(rdt + (1-H_t)\delta(1-\Delta)dt - (1-H_{t^-})\zeta dM_t^Q\big), \end{aligned} \tag{5}$$

where

$$p(t,T_1) = \begin{cases} e^{-(r+\delta)(T_1-t)}, & \text{if } \tau > t, \\ (1-\zeta)e^{-(r+\delta)(T_1-t)}e^{(t-\tau)}, & \text{if } \tau \leq . \end{cases}$$

The instantaneous expected return of the defaultable bond, given in equation (5), can be seen to consist of two components similar to what Yu (2002) argues (see equation (15) in Yu (2002). The first component is the return on an otherwise identical default-free bond. The second is the difference between the risk neutral credit spread and the real world credit spread provided that default has not already occurred by time t. The implication of Jarrow's et al. (2005) argument is that the second component will disappear. In order to see this clearly we explicitly express the difference of the credit spreads in terms of the intensities under Q and P such as:

$$\delta - \delta\Delta = \zeta h^Q - \zeta\Delta h^Q = \zeta h^Q - \zeta h^P = \big(\frac{h^Q}{h^P} - 1\big)\zeta h^P,$$

where $\delta^P = \zeta h^P$ is the credit spread under the real world probability measure. Therefore, if h^Q is equal to hP then the second term in equation (5) vanishes provided that default has not already occurred by time t. In such a case, the investor requires no compensation for bearing the risk of the default event. Through simple parameterization of the default event risk we can show the difference of the credit spreads in terms of the default event risk premium, namely,

$$\delta - \delta\Delta = \big(\frac{h^Q}{h^P} - 1\big)\zeta h^P = \big(\frac{1}{\Delta} - 1\big)\delta^P,$$

where $\frac{1}{\Delta} = \frac{h^Q}{h^P}$. Thus the compensation for the default event risk premium becomes $\big(\frac{1}{\Delta} - 1\big)$. We can also express the dynamics of stock price under P as follows:

$$dS_t = S_t\big(\mu dt + \sigma dW^P\big). \tag{6}$$

The stock price grows at a instantaneous rate of μ and has the instantaneous volatility of under the real world probability measure P. Note that we change the drift term of the dynamics of stock price, but do not change the volatility of it through the change of risk neutral probability measure to real world probability measure. On the other hand, the dynamics of a savings account has the same representation under both measure Q and P since it is non-stochastic.
The pricing dynamics of each security under the real world probability measure P such as equation (3), (5), and (6) are utilized in the next section.

3 Optimal portfolio problems using the utility function

We use stochastic optimal control theory to find the optimal allocations among a savings account, a stock (or stock index), and a defaultable bond for the investor whose objective is to maximize her conditional expected utility of the terminal wealth over the investment horizon $[0, T]$. We assume throughout that $T < T_1$, where T_1 is the time of maturity of the defaultable bond. We consider that short rate of interest is constant and non-negative.
To start, let N_t^s be the number of shares of stocks the investor longs $N_t^s \geq 0$ or shorts $N_t^s < 0$ at time t. Similarly, N_t^P and N_t^B are the number of shares of the defaultable bond and a savings account she buys or sells at time t for all $t \in [0, T]$. Then the process $N = \big(N^s, N^p, N^B\big)$ is called a portfolio process. Now we define $K_t(N)$ as the wealth of the portfolio process $N = \big(N^s, N^P, N^B\big)$ at time t, that is

$$K_t(N) = N_t^s S_t + N_t^B B_t + N_t^P p(t, T_1), \qquad t \in [0, T].$$

The wealth of the portfolio process N starting with $\nu > 0$ at time t will be denoted as $K_t^\nu(N)$. Now, let $\big(\pi_t^s, \pi_t^p, \pi_t^B\big)$ be defined as

$$\pi_t^s = \frac{N_t^s S_t}{K_{t^-}(N)}, \qquad \pi_t^P = \frac{N_t^P P(t^-, T_1)}{K_{t^-}(N)}, \qquad \pi_t^B = \frac{N_t^B B_t}{K_{t^-}(N)}.$$

Note that $\pi_t = \big(\pi_t^s, \pi_t^P, \pi_t^B\big)$ represents the fraction of the wealth invested in each asset at time t and is called fractional strategy or simply strategy in the sequel. For all $t \in [0, T]$ we shall consider the following wealth equation:

$$\begin{aligned} dK_t(\pi) &= K_{t^-}(\pi)\big(\big(\pi_t^s \mu + \big(\delta(1 - H_t)(1 - \Delta)\pi_t^p + r\big) + r(1 - \pi_t^s - \pi_t^p)\big)dt \\ &\quad + \pi_t^s \sigma dW_t^P - \pi_t^p \zeta dM_t^P\big), \end{aligned} \tag{7}$$

with $K_0(\pi) = \nu$, where $dM_r^P = (1 - H_t)dM_r^P$, and $h^P = h^P\Delta$.

3.1 Optimization Problems

We consider the following joint dynamics of the wealth equation $K_t(\pi)$ given in (7) and for the factor equation H_t for $t \in [0, T]$,

$$\begin{aligned} dK_t(\pi) &= K_{t^-}(\pi)\big(\big(\pi_t^s \mu + \big(\delta(1 - H_t)(1 - \Delta)\pi_t^p + r\big) + r(1 - \pi_t^s - \pi_t^p)\big)dt \\ &\quad + \pi_t^s \sigma dW_t^P - \pi_t^p \zeta dM_t^P\big), \end{aligned} \tag{8}$$

where $dH_t = h^P(1-H_t)dt + dM_t^P$, with $K_0(\pi) = \nu$, $H_0 = 0$. Moreover, we use $h^P = h^P\Delta$ and $\delta = h^P\zeta$.
To present the optimization problem, we consider the utility function as

$$U(x) = \frac{x^\gamma}{\gamma}, \qquad 0 < \gamma < 1, \;\; x \geq 0. \tag{9}$$

We shall study the optimization problems in case of the given utility function (9) such as:

$$O(t,\nu,z) = \sup_{(\pi^s,\pi^p)\in\Pi_t(G)} E_P\big(\frac{K_T^\gamma(\pi)}{\gamma}\big|K_t(\pi) = \nu, H_t = z\big), \tag{10}$$

for all $(t,\nu,z) \in (0,T)\times(0,\infty)\times\{0,1\}$ over all admissible strategies π. Let us define the following processes as

$$\pi_t^{s*} = \frac{(\mu - r)}{\sigma^2(1-\gamma)}, \qquad t \in [0,T],$$

$$\pi_t^{p*} = \begin{cases} \frac{1}{\zeta}\big(1 - \big((\frac{1}{\Delta}\big)^{\frac{1}{1-\gamma}} e^{\big(-\frac{h^P}{1-\gamma} - \frac{h^P}{\Delta}\frac{\gamma}{1-\gamma}\big)(T-t)} + e^{\big(-\frac{h^P}{1-\gamma} - \frac{h^P}{\Delta}\frac{\gamma}{1-\gamma}\big)(T-t)}, & \text{if} \quad 0 \leq t < \tau\wedge T, \\ 0, \quad \text{if } t \in \big((\tau\wedge T), T\big), \end{cases}$$

and

$$\pi_t^{B*} = \begin{cases} 1 - \pi_t^{s*} - \pi_t^{p*}, & \text{if} \quad 0 \leq t < \tau\wedge T, \\ 1 - \pi_t^{s*}, & \text{if} \quad t \in \big((\tau\wedge T), T\big). \end{cases}$$

Then the process π defined above is an optimal strategy. Now we are ready to state the optimality theorem.

Theorem 3.1. *(Optimal strategies of risk model). The process π^* defined above is an optimal strategy.*

Note that the optimal trading strategies are independent of the wealth since the utility fuction is a homogeneous function of wealth. Next we derive the optimal value function $O(.,.,.)$ This is often called the indirect utility function since it measures the highest attainable expected utility the investor can derive from her current wealth in the current state of the world.

Theorem 3.2. *(Indirect utility function). Given the utility function (9), we obtain the optimal objective function $O(t,\nu,z)$ as*

$$O(t,\nu,z) = \begin{cases} \big(f(t)\big)^{1-\gamma)\times\big(\frac{\nu^\gamma}{\gamma}\big)}, & \text{if} \quad z = 0, \\ e^{\big(\frac{1}{2}\frac{\gamma}{1-\gamma}\frac{(\mu-r)^2}{\sigma^2} + r\gamma\big)(T-t)} \times \big(\frac{\nu^\gamma}{\gamma}\big), & \text{if } z = 1, \end{cases}$$

where

$$\begin{aligned} f(t) \;=\; & e^{\big(\frac{1}{2}\frac{\gamma}{(1-\gamma)^2}\frac{(\mu-r)^2}{\sigma^2} + \frac{r\gamma}{1-\gamma}\big)(T-t)} \\ & \times\big(e^{\big(-\frac{h^P}{1-\gamma} - \frac{h^P}{\Delta}\frac{\gamma}{1-\gamma}\big)(T-t)} + \big(e^{\big(-\frac{h^P}{1-\gamma} - \frac{h^P}{\Delta}\frac{\gamma}{1-\gamma}\big)(T-t)} - e^{-\frac{r\gamma^2}{1-\gamma}(T-t)}\big) \\ & \times h^P\big(\frac{1}{\Delta}\big)^{\frac{1}{1}-\gamma}\frac{1-\gamma}{h^P(\gamma-\Delta) + r\Delta\gamma^2}. \end{aligned}$$

Proof. In the following, we prove that $O(t) \geq O(t,\nu,z)$ for all $\nu \in S_t$, and $O_1(t) = O_1(u,\nu^*,z^*)$. Using the quasi variational inequality, for any $q > 0$ we have

$$-\eta O\big(R^{(\nu,z^*)}(t)\big) + K^{(e_1,e_2^*)}O\big(R^{(\nu,z^*)}(t)\big)\big) \leq -q\big(R^{(\nu,z^*)}(t)\big).$$

On the other hand, since $-\alpha_2 \leq O'(t) \leq \alpha_1$, therefore for $z \in \Delta_1$ and $z \leq t$, the inequality

$$O\big(R^{(\nu,z^*)}(t)\big) - O\big(R^{(\nu,z^*)}(t-)\big) \leq \alpha_1\big(A_1^{s_1}(z) - A_1^\nu(t-)\big),$$

holds, and for $z \in \Delta_2$, we have

$$O\big(R^{(\nu,z^*)}(z)\big) - O\big(R^{(\nu,z^*)}(z-)\big) = \alpha_2\big(A_2^{s_2^*}(t) - A_2^{z^*}(t-)\big).$$

On the other hand, we obtain

$$\begin{aligned} O(t) \quad \geq \quad & E\big[\int_0^t e^{-\eta t} q_1\big(R^{(\nu,z^*)}\big)dt - \alpha_1 \int_0^t e^{-\eta t} dA_1^{\nu}(t) \\ & +\alpha_2 \int_0^t e^{-\eta t} dA_2^{z^*}(t) \\ & +e^{-\eta z} O\big(R^{(\nu,z^*)}(t)\big)\big] = O(t,\nu,z^*). \end{aligned}$$

By the same procedure, we will obtain that $O(t) \geq O(t,\nu,z^*)$ and this completes the proof. □

This Theorem shows that $O_\nu = \frac{\partial}{\partial \nu} O(t,\nu,z) > 0$ and $O_{\nu\nu} = \frac{\partial^2}{\partial \nu^2} O(t,\nu,z) < 0$ This indicates that $O(t,\nu,z)$ is an concave function of initial wealth ν.

4 Implication for asset allocation

We investigate the implications of the optimal strategies.

4.1 Optimal strategy for a stock

The optimal strategy for the stock is constant irrespective of pre-default or post-default. The reason is that the stock has no correlation with the defaultable bond, which means there is no need to hedge for the default. Thus the optimal strategy for the stock is invariant to the default event risk.
Let us decompose the optimal strategy π^{s*} for the stock as follows:

$$\pi^{s*} = -\frac{O_\nu}{\nu O_{\nu\nu}} \frac{\mu - r}{\sigma^2} = -\frac{1}{\gamma - 1} \frac{\mu - r}{\sigma^2}, \tag{11}$$

The optimal strategy π^{s*} for the stock is interpreted as the product of the relative risk tolerance $-\frac{O_\nu}{\nu O_{\nu\nu}}$ (i.e. the inverse of the relative risk aversion) of the indirect utility function and market price of equity risk (or Sharpe ratio of a stock) $\frac{\mu - r}{\sigma}$ normalized by the volatility of the stock return σ. Thus the demand for equity contains only so-called "myopic" term despite the fact that the investment opportunity set is stochastic in this economy. s usual, the myopic demand for the stock decreases as the risk aversion coefficient γ is close to zero. This is reasonable since close to zero indicates that the relative risk aversion $1 - \gamma$ of the investor whose utility function is close to one.

4.2 Optimal strategy for a bond

It is clear from Lemma 2.4 that the drift and volatility of the return of the defaultable bond changes discretely at default. That is, the stochastically evolving default process H, which is the only risk source of randomness for the defaultable bond, is captured in the drift and volatility of the return of the defaultable bond. We can interpret the change that occurs at default as a change in the investment opportunity set of the investor. In the default literature it is commonly assumed that the investor knows whether the default event has occurred or not. Thus we can simply analyze the optimal defaultable bond investment on the pre and post default space. From the analytical result it is clear that the pre-default optimal strategy of the defaultable bond depends on the investment planning horizon. The investor allocates more when she has longer planing horizon.
It can be easily shown that the probability of default in our framework is greater for longer planning horizon. Nevertheless, we find that the investor allocates greater amount of wealth the longer the amount of time remaining. Although this result at first appears to be perverse it can be explained in terms of the default risk premium. Recall from Section 2 that if default follows a Poisson process then the pre-default price of the defaultable bond is given $p(t,T_1) = e^{(r+\delta)(T_1 - t)}$, where $T < T_1$. We can see that as time t approaches to T, both

the investment horizon $T-t$ and the time to maturity of the defaultable bond T_1-t tend to zero. This leads to an increase in the price of the defaultable bond providing no default has yet to occur. In other words, the defaultable bond is getting relatively cheap as the investment horizon increases. Since the default event risk premium $\frac{h^Q}{h^P}$ is assumed to be the same throughout the planning horizon and the time to maturity then the only risk the investor faces with regards to her return in the defaultable bond is the default itself. Thus the investor would naturally invest greater amount at longer planning horizon since for given probability of default the bond naturally appreciates in value as it achieves par value at maturity providing there is no default. Following the buy low sell high rule, the investor should invest more in the defaultable bond when its price is low, that is, for longer maturity date, all else equal.

5 Sensitivity analysis on the model parameters

In this section we analyze the behavior of the optimal strategy of the defaultable bond in particular. Since the optimal strategy of the defaultable security is a function of the default risk premium, the loss rate, the time to maturity, the risk aversion, the default intensity, and the interest rate. We quantify the optimal investment for the defaultable bond by adopting parameter values from the literature. Hou (2003) and Driessen (2005) also provide parameter values. Since a higher default risk premium induces a positive amount of investment in the defaultable security, we are not surprised to find a positive relationship between optimal investment in the defaultable security and the default risk premium. Figure 1 shows that the investor buys more defaultable security as the default risk premium increases. Note that the optimal investment in the defaultable bond is increasing with a decreasing rate when the default risk premia increase.

In Figure 1, we also observe that if the loss rate is small then the investor responds more to the increase in the default risk premium. As the loss rate increases, however, the investor still enjoys the high risk premium, but with much care. The concave shape of optimal investment of the defaultable security is getting flatter as ζ increases. When the loss rate (recovery rate) is very low (high), the marginal increase of the optimal strategy of the defaultable bond is huge. Thus the investor enjoys the priced default event risk premium since she can recover a lot even in the case of default. Note that if the default risk premium is equal to one the investor does not invest in the defaultable security, which is shown as the horizontal line.

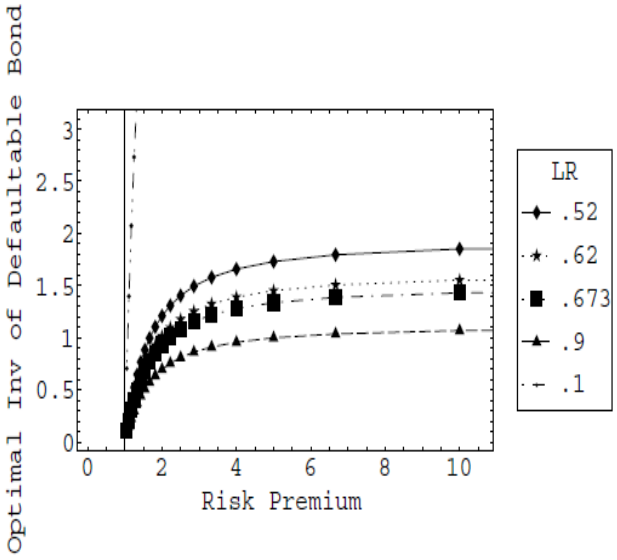


Figure 1: Optimal investment of a bond versus the risk premium is 1, 0.52, 0.62, 0.673 and 0.9 given $\gamma = 0.5$.

Next we analyze the relationship between the loss rate and the optimal strategy. Since the higher loss rates mean less recovery values the optimal investment in the defaultable security shows a decaying pattern. Figure 2 indicates that when the default risk premium is high the investor decreases the investment in the defaultable security, but still invests more compared to the case of low default risk premium as the loss rate increases.

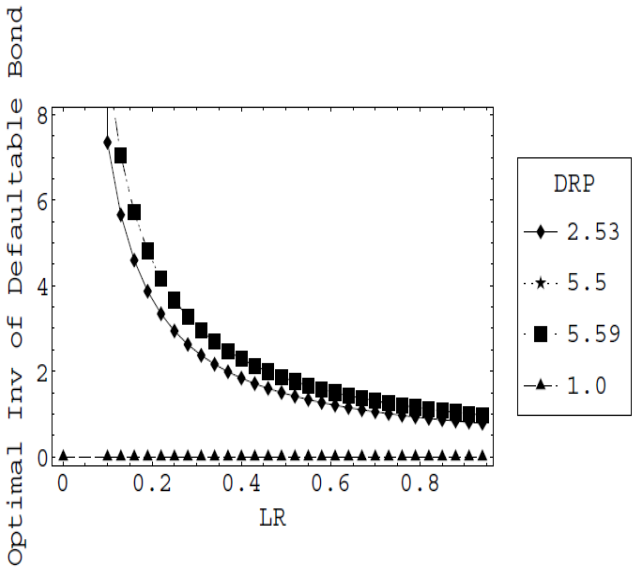


Figure 2: Optimal investment of a defaultable bond versus the loss rate when risk premium is 1, 2.53, 5.5, 5.59 given $\gamma = 0.5$.

Discussion and Results

In this paper we address the optimal portfolio problem with a defaultable security. We adopt the intensity based approach to modeling defaultable securities. In addition, we assume the recovery of market value throughout the paper. This is a key distinguishing feature, as most papers ignore the problem of a positive recovery value in the event of a default or they just simply assume zero recovery. Thus this paper improves upon the existing literature by explicitly modeling the recovery amount in the dynamics of the defaultable bond price. The theory of stochastic optimal control is utilized as opposed to the martingale approach to solve the investor's problem. We obtained closed form solutions for the optimal strategies for a default-free bank account, a stock, and a defaultable bond.

Acknowledgment

The author would like to thank the reviewers for providing insightful comments that improved the presentation of this paper.

References

Aydoğan, B. and Steffensen, M. (2024), *Optimal investment strategies under the relative performance in jump-diffusion markets*, Decisions in Economics and Finance, https://doi.org/10.1007/s10203-024-00499-1.

Bäuerle, N. and Göll, T. (2024), *Nash equilibria for relative investors with (non)linear price impact*, Mathematics and Financial Economics, 18, 27-48.

Deng, C., Su, X. and Zhou, C. (2024), *Relative wealth concerns with partial information and heterogeneous priors*, SIAM Journal on Financial Mathematics, 15(2), 360-398.

Dickson, D. C. M. (2005), *Insurance Risk and Ruin*, Cambridge: Cambridge University Press.

Driessen, J. (2005), *Is Default Event Risk Priced in Corporate Bonds?*, Review of Financial Studies, 18, 165-195.

Hou, Y. and Jin, X. (2002), *Optimal Investment with Default Risk*, FAME research paper **46**, Switzerland.

Hou, Y. (2003), *Integrating Market Risk and Credit Risk: A Dynamic Asset allocation perspective*, Working Paper, Switzerland. Yale University.

Kraft, H. (2020), *Meyer-Wehmann, A., Seifried, F.T.: Dynamic asset allocation with relative wealth concerns in incomplete markets*, Journal of Economic Dynamics and Control, 113, 103857.

Korn, R. and Kraft, H. (2003), *Optimal Portfolios with Defaultable Securities: A Firm value Approach*, International Journal of Theoretical and Applied Finance 6, 793-819.

Lacker, D., Zariphopoulou, T. (2019), *Mean-field and n-agent games for optimal investment under relative performance criteria*, Mathematical finance, 29, 1003-1038.

Walder, R. (2002), *Dynamic Allocation of Treasury and Corporate Bond Portfolios*, FAME research paper, Switzerland.

Zhu, H., Cao, M. and Zhang, C. (2019), *Time-consistent investment and reinsurance strategies for meanvariance insurers with relative performance concerns under the heston model*, Finance Research Letters, 30, 280-291.

Zariphopoulou, T. (2024), *Mean field and n-player games in itô-diffusion markets under forward performance criteria*, Probability, Uncertainty and Quantitative Risk, 9(2), 123-148.