Controllability on the infinite-dimensional group of orientation-preserving diffeomorphisms of the unit circle

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Abstract. In this paper, we give a generalization of Chow–Rashevsky's theorem for control systems in regular connected manifolds modeled on convenient locally convex vector spaces which are not necessarily normable. To indicate an application of our approach to the infinite-dimensional geometric control problems, we conclude with a novel controllability result on the group of orientation-preserving diffeomorphisms of the unit circle, which has applications in, e.g., conformal field theory as well as string theory and statistical mechanics.

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1. Introduction

Control theory is in fact the theory of prescribing motion for dynamical systems rather than describing their observed behavior. One of the fundamental problems in control theory is that of controllability, the question of whether one can drive the system from one point to another with a given class of controls. A classical result in geometric control theory of finite-dimensional (nonlinear) systems is Chow–Rashevsky's theorem that gives a sufficient condition for controllability on any connected manifold of finite dimension. In other words, the classical Chow–Rashevsky's theorem, which is in fact a primary theorem in subriemannian geometry, gives a global connectivity property of a subriemannian manifold. The classical result was proved independently and almost simultaneously by Chow [2] and Rashevsky [6].

One of the main results of this paper gives a generalization of the above classical result to the case of *infinite*-dimensional manifolds, which makes it possible to consider even more general classes of "controllable" nonlinear systems –cf. Theorem 2.3. The

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approach we follow here is based on the ones initiated in the works of Jurdjevic, Agrachev, Sachkov, Kriegl and Michor; for a through treatment, we refer the reader to [3], [4] and the references given there. As a corollary of Theorem 2.3, we conclude our paper with a novel controllability result on the group of orientation-preserving diffeomorphisms of the unit circle (see Subsection 2.2).

2. Main results

To introduce the notion of *completeness* in infinite-dimensional locally convex vector spaces, we need the following result that states when we call a vector space *convenient*.

LEMMA 2.1. *Let E be a locally convex vector space. E is said to be* convenient *if one of the following equivalent (completeness) conditions is satisfied:*

(1) *Every Mackey-Cauchy net converges in E; i.e., E is Mackey complete.*
(2) *Every Mackey-Cauchy sequence converges in E.*
(3) *If $M \subset E$ is absolutely convex closed bounded, then E_M is a Banach space.*
(4) *For every bounded set $M \subset E$ there exists an absolutely convex bounded set $M' \supseteq M$ such that $E_{M'}$ is a Banach space.*

The key to formulating the main results of this paper is the following lemma.

LEMMA 2.2. *Let E be a convenient real locally convex vector space, and $B \subset E$ be a closed nonempty proper subset. Then there exists a boundary element $a_* \in B$, an open set $U \subset E$ containing a_*, and a solid cone*

$$C_{a_*} := \{a_* + t\,(x - a_*) \mid x \in \mathcal{X},\ t \geq 0\},$$

such that $U \cap C_{a_*} \cap B = \{a_*\}$, where $\mathcal{X} \subset E$ is some convex closed set.

2.1. A Chow–Rashevsky theorem. In our study of control systems, we always assume that the state space M is a smooth manifold modeled on a locally convex space. In what follows, $\mathcal{F} \subset \mathrm{Vec}\,(M)$ stands for any family of complete smooth vector fields. Thus each element $X \in \mathcal{F}$ generates a one-parameter group of diffeomorphisms $\left\{\mathrm{e}^{tX} \,|\, t \in \mathbb{R}\right\}=$ flow of X in M. Let $\mathcal{P}\,(\mathcal{F}) =: \mathcal{P}$ denote the group of diffeomorphisms of M generated by flows $\left\{\mathrm{e}^{tX} \,|\, t \in \mathbb{R}\right\}_{X \in \mathcal{F}}$ of $\mathcal{F}$. Each element Φ of $\mathcal{P} \subset \mathrm{Diff}\,(M)$ is of the form

$$\Phi = \mathrm{e}^{t_k X_k} \circ \mathrm{e}^{t_{k-1} X_{k-1}} \circ \cdots \circ \mathrm{e}^{t_1 X_1},$$

for some $k \in \mathbb{N}$, $t_1, \ldots, t_k \in \mathbb{R}$ and some vector fields $X_1, \ldots, X_k \in \mathcal{F}$. In fact $\mathcal{P}\,(\mathcal{F}) = \mathcal{P}$ acts on M in the obvious way and partitions M into the sets $\mathcal{P}\,(x) = \{\Phi\,(x) \,|\, \Phi \in \mathcal{P}\}$ for x in M.

Since the set $\mathrm{Vec}\,(M)$ of all smooth vector fields on M has the structure of a real Lie algebra under the Lie-bracket operation, to the given $\mathcal{F} \subset \mathrm{Vec}\,(M)$ there corresponds the Lie subalgebra $\mathrm{Lie}\,\mathcal{F}$ of $\mathrm{Vec}\,(M)$ generated by $\mathcal{F}$. The evaluation of $\mathrm{Lie}\,\mathcal{F}$ at $x \in M$ will be denoted by $\mathrm{Lie}_x\,\mathcal{F} = \{V\,(x) \,|\, V \in \mathrm{Lie}\,\mathcal{F}\}$.

One of our aims, in this paper, is to prove a generalization of Chow–Rashevsky's theorem for control systems in *regular* connected manifolds M modeled on convenient (infinite-dimensional) locally convex spaces E. We call here a smooth manifold *regular* if any neighborhood of any point $a \in M$ contains the closure of some smaller neighborhood of the same point a in M. The regularity condition is in fact satisfied if, for example, M is locally compact or is a topological group [5]. This result can be formulated as follows.

Theorem 2.3. *Let M be a regular connected manifold modeled on a convenient locally convex space E, and $\mathcal{F}$ be a family of smooth vector fields on M. If $\mathrm{Lie}_x\,\mathcal{F}$ is dense in T_xM for all x in M, then $\mathcal{P}\left(x\right)$ is dense in M for all $x \in M$.*

The proof of Theorem 2.3 is based on a corollary of Lemma 2.2, to be given as follows, whose formulation requires the next definition.

Definition 2.4. Let $\mathcal{B}$ be an arbitrary subset of the manifold M. For any $x \in \mathcal{B}$ and $\nu \in T_xM$, we say that ν is *tangent to $\mathcal{B}$ at x* if there exists a curve $\gamma_\nu : [0,1] \to M$ such that $\gamma_\nu\left(0\right) = x$, $\dot{\gamma}_\nu\left(0\right)$ exists and is equal to ν, and $\gamma_\nu\left(t\right) \in \mathcal{B}$ for all t. We denote by $\mathcal{T}_x\mathcal{B}$ the set of all tangent vectors to $\mathcal{B}$ at x.

Then, as a result of Lemma 2.2, we have

Corollary 2.5. *Let M be a regular connected manifold modelled on a convenient locally convex space E, and $\mathcal{B} \subseteq M$ be a closed nonempty subset. If $\mathcal{T}_x\mathcal{B}$ is dense in T_xM for every $x \in \mathcal{B}$ then $\mathcal{B} = M$.*

Accordingly, *a sketch* of the proof of Theorem 2.3 can be given as follows. We first prove the following two claims that, on one hand, for any family $\mathcal{F} \subset \mathrm{Vec}\left(M\right)$, we have

$$\mathrm{Lie}_x\,\mathcal{F} \subseteq \mathcal{T}_x\mathcal{P}\left(x\right), \quad \text{for every } x \in M,$$

and, on the other hand, we get

$$\mathcal{P}\left(x\right) \subset \overline{\mathcal{P}\left(p\right)}, \quad \text{for any } p \in M,\ x \in \overline{\mathcal{P}\left(p\right)}.$$

Then, by the above-mentioned two relations, we derive $\mathrm{Lie}_x\,\mathcal{F} \subseteq \mathcal{T}_x\mathcal{P}\left(x\right) \subset \mathcal{T}_x\overline{\mathcal{P}\left(p\right)} \subset T_xM$ for any $p \in M$ and $x \in \overline{\mathcal{P}\left(p\right)}$, and consequently

$$T_xM = \overline{\mathrm{Lie}_x\,\mathcal{F}} = \overline{\mathcal{T}_x\overline{\mathcal{P}\left(p\right)}}.$$

Therefore, the theorem follows from Corollary 2.5 with $\mathcal{B} = \overline{\mathcal{P}\left(p\right)}$.

We conclude our paper by showing as to how Theorem 2.3 works on the group of orientation-preserving diffeomorphisms of the unit circle, which has applications in, e.g., conformal field theory as well as string theory and statistical mechanics.

2.2. Controllability on $\mathrm{Diff}_0\left(S^1\right)$. Let S^1 be the unit circle embedded into the Euclidean space $\mathbb{R}^2$, and denote by $M = \mathrm{Diff}_0\left(S^1\right)$ the identity connected component of the group of diffeomorphisms of S^1. In fact M is a Lie group modeled on the locally convex space $\mathrm{Vec}\left(S^1\right)$, cf. [5, pp. 1039–1041]. Hence the tangent space of M at $\mathrm{id} \in M$ can be identified with

$$T_{\mathrm{id}}M = \mathrm{Vec}\left(S^1\right) = \left\{\nu\left(\theta\right)\partial_\theta \mid \theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z},\ \nu \in C^\infty\left(S^1, \mathbb{R}\right)\right\},$$

where ∂_θ stands for $\frac{\partial}{\partial\theta}$. Under this identification, the commutator of two elements in the Lie algebra $\mathrm{Vec}\left(S^1\right)$ of smooth vector fields on the circle is given by

$$\left[\nu\left(\theta\right)\partial_\theta, \omega\left(\theta\right)\partial_\theta\right] = \left(\nu'\left(\theta\right)\omega\left(\theta\right) - \omega'\left(\theta\right)\nu\left(\theta\right)\right)\partial_\theta,$$

where ν' denotes the θ-derivative of ν. Note that this Lie bracket is the negative of the commonly assumed commutator of vector fields. It is worth pointing out that the space of all functions $\nu \in C^\infty\left(S^1, \mathbb{R}\right)$ is in fact a Fréchet space with the countable base of seminorms $\{p_0, p_n \,|n \in \mathbb{N}\}$ where $p_0\left(\nu\right) = \max_{\theta \in S^1}\left|\nu\left(\theta\right)\right|$, and $p_n\left(\nu\right) = \max_{\theta \in S^1}\left|\frac{d^n}{d\theta^n}\nu\left(\theta\right)\right|$ for any $n \in \mathbb{N}$. Therefore $\mathrm{Vec}\left(S^1\right)$ is a real Fréchet space, and hence M is a convenient Lie group (see [5] for more details). Let $\mathrm{Vec}\left(S^1\right)_{\mathbb{C}} = \mathrm{Vec}\left(S^1\right) \otimes \mathbb{C}$ be the complexification of the Lie algebra $\mathrm{Vec}\left(S^1\right)$. An element $\nu(\theta)\,\partial_\theta \in \mathrm{Vec}\left(S^1\right)_{\mathbb{C}}$ can in fact be expressed using the Fourier expansion of $\nu(\theta) = \sum_{n=-\infty}^{+\infty} a_n \mathrm{e}^{in\theta}$, where $a_n \in \mathbb{C}$ and $\mathrm{e}^{in\theta} = \cos n\theta + i \sin n\theta$. Hence $B_{\mathrm{id}} := \{\partial_\theta, \cos n\theta\,\partial_\theta, \sin n\theta\,\partial_\theta\}_{n=1}^{\infty}$ forms a basis for $T_{\mathrm{id}}M = \mathrm{Vec}\left(S^1\right)$. Let $\widetilde{B}_{\mathrm{id}} = \{\cos\theta\,\partial_\theta, \sin\theta\,\partial_\theta, \cos 2\theta\,\partial_\theta, \sin 2\theta\,\partial_\theta\} \subset B_{\mathrm{id}}$. It is easily seen that

$$[\sin\theta\,\partial_\theta, \cos\theta\,\partial_\theta] = \partial_\theta,$$

$$\left[i\mathrm{e}^{in\theta}\partial_\theta, i\mathrm{e}^{im\theta}\partial_\theta\right] = (m-n)\, i\mathrm{e}^{i(m+n)\theta}\partial_\theta.$$

Comparing the real and imaginary parts of both sides of the latter equality, we deduce that taking linear combinations of all possible (iterated) Lie brackets of elements in $\widetilde{B}_{\mathrm{id}}$ one can generate all vector fields in B_{id}; e.g.,

$$\sin 3\theta\,\partial_\theta = [\cos\theta\,\partial_\theta, \cos 2\theta\,\partial_\theta] - [\sin\theta\,\partial_\theta, \sin 2\theta\,\partial_\theta]\,,$$

$$\cos 3\theta\,\partial_\theta = -\left([\sin\theta\,\partial_\theta, \cos 2\theta\,\partial_\theta] + [\cos\theta\,\partial_\theta, \sin 2\theta\,\partial_\theta]\right),$$

$$\sin 4\theta\,\partial_\theta = \left([\cos\theta\,\partial_\theta, \cos 3\theta\,\partial_\theta] - [\sin\theta\,\partial_\theta, \sin 3\theta\,\partial_\theta]\right)/2, \quad \text{etc.}$$

Let us now consider $\widetilde{B}_\phi := d_{\mathrm{id}}R_\phi\left(\widetilde{B}_{\mathrm{id}}\right) \subset T_\phi M$, where $\phi \in M = \mathrm{Diff}_0\left(S^1\right)$, and $R_\phi : M \ni \psi \longmapsto \psi \circ \phi \in M$ is the right translation map. Accordingly, we can define the distribution $\mathcal{H} = \bigsqcup_{\phi\in M} \mathcal{H}_\phi \subset TM$ where $\mathcal{H}_\phi := \mathrm{span}\,\widetilde{B}_\phi \subset T_\phi M$. Setting $\mathcal{F} := \{X \in C^\infty(M, \mathcal{H}) \mid X(\phi) \in \mathcal{H}_\phi \text{ for any } \phi \in M\}$, we conclude that

$$\mathrm{Lie}_\phi\,\mathcal{F} = T_\phi M, \quad \text{for any } \phi \in M,$$

due to the definition of $\mathcal{H}_\phi$ and the construction of $\widetilde{B}_\phi$. Theorem 2.3 now shows that $\overline{\mathcal{P}(\mathcal{F})(\phi)} = M$ for any $\phi \in M$.

It is worth noting that our controllability result on the group of diffeomorphisms $\mathrm{Diff}_0\left(S^1\right)$, given in this section, does not follow from those obtained by Agrachev and Caponigro [1].

3. Conclusion

We have proved a generalization of Chow–Rashevsky's theorem for control systems in regular connected manifolds modeled on convenient not-necessarily-normable spaces; as an application of this theorem, we have concluded our paper with a novel controllability result on the group of orientation-preserving diffeomorphisms of the unit circle.

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