A note on MV-pseudo norm

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Abstract. In this paper, we define the notions of MV-pseudo norm and MV-pseudo metric on MV-algebras and study some of their algebraic properties. The notion of uniform MV-algebra is also introduced and its relationship to MV-pseudo metrics is studied.

Keywords: MV-algebra, MV-pseudo norm, MV-pseduo metric, Uniform MV-algebra.

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1. Introduction

MV-algebras, which were introduced by Chang in [**?**] in 1958, prove the completeness theorem for $\aleph_0$-valued Lukasiewicz logic. Our aim in this article is to introduce and study *MV-pseudo metrics* on MV-algebras. To this end, we first define MV-pseudo norms on MV-algebras, and study their algebraic properties.

The article is organized as follows: in Section 2 we present some definitions and results of the MV-algebra theory and uniform spaces which will be used later in the paper.

In Section 3 we define the concept of MV-pseudo norm, and discuss its algebraic properties and its relation to filters and ideals. Also, the relationship between MV-pseudo norm on MV-algebras and qoutient MV-algebras will be examined in this section. Finally, we show that if $f : A_1 \to A_2$ is an isomorphism between MV-algebras, and N_{A_1} is an MV-pseudo norm on A_1, then $N_{A_2} = N_{A_1} \circ f^{-1}$ is an MV-pseudo norm on A_2.

In Section 4, we define MV-pseudo metrics and examine their relations to MV-pseudo norms. There are also a few theorems about the relationship between MV-pseudo metrics and uniform MV-algebras. Theorem **??** in particular provides an efficient way to construct an MV-pseudo metric on MV-algebras.

1.1. MV-algebras. An *MV-algebra* is an algebra $(A, \oplus, *, 0)$ of type $(2, 1, 0)$ such that for every $x, y \in A$,
$(M1)$ $(A, \oplus, 0)$ is a commutative monoid,
$(M2)$ $x \oplus 0^* = 0^*$,

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$(M3)$ $(x^*)^* = x$, and
$(M4)$ $(x^* \oplus y)^* \oplus y = (x \oplus y^*)^* \oplus x$. [?]
In an MV-algebra A, for every $x, y \in A$, define
$(M5)$ $1 := 0^*$;
$(M6)$ $x \odot y := (x^* \oplus y^*)^*$;
$(M7)$ $x \ominus y := x \odot y^*$;
$(M8)$ $x \to y := (x \odot y^*)^*$;
$(M9)$ $x \rightsquigarrow y := (x \oplus y^*)^*$.

In an MV-algebra A, for every $x, y \in A$, we write $x \leq y$ if and only if $x^* \oplus y = 1$. It is well-know that $\leq$ is a partial order on A, which gives A the structure of a distributive lattice, where the join and meet are defined by $x \wedge y = y \odot (y^* \oplus x)$ and $x \vee y = x \oplus (y \ominus x)$, respectively, 0 is the least element and 1 is the greatest element. By $(M6)$ and $(M7)$, for every $x, y \in A, x \leq y \Longleftrightarrow x \ominus y = 0$.

Definition 1.1. Let A be an MV-algebra.
(1) A non-empty subset I of A is called an *ideal* if it satisfies the following conditions.
(I1) For every $x, y \in I, x \oplus y \in I$.
(I2) If $x \in I$ and $y \leq x$, then $y \in I$. [?]
(2) A non-empty subset F of A is called a *filter* if it satisfies the following conditions.
(F1) For every $x, y \in F, x \odot y \in F$.
(F2) If $x \in F$ and $x \leq y$, then $y \in F$. [?]

Proposition 1.2. *[?] Let F be a filter and I be an ideal of an MV-algebra A. Then the following are congruence relations on A.*

$$x \overset{F}{\equiv} y \iff x \to y \in F \text{ and } y \to x \in F.$$
$$x \overset{I}{\equiv} y \iff x \ominus y \in I \text{ and } y \ominus x \in I.$$

Moreover, if $x/F = \{y \in A : x \overset{F}{\equiv} y\}$, $A/F = \{x/F : x \in A\}$, $x/I = \{y \in A : x \overset{I}{\equiv} y\}$ and $A/I = \{x/I : x \in A\}$, then both A/F and A/I are quotient MV-algebras with the operations

$$x/F \odot y/F = (x \odot y)/F,\ x/I \oplus y/I = (x \oplus y)/I,\ (x/F)^* = x^*/F \text{ and } (x/I)^* = x^*/I.$$

2. MV-pseudo norms on MV-algebras

Definition 2.1. Let A be an MV-algebra. Then, we say that a map $N : A \longrightarrow \mathbb{R}$ is an *MV-pseudo norm* on A if the following hold.
$(N1)$ $N(x \oplus y) \leq N(x) + N(y)$.
$(N2)$ $N(x^*) \leq N(1) - N(x)$.
An MV-pseudo norm is an *MV-norm* if $N(x) = 0 \Leftrightarrow x = 0$.

Example 2.2. Let X be a finite set and $(P(X), \cup, *, \varnothing, X)$ be the MV-algebra in which for each $B \in P(X)$, B^* is the complement of B in X. The map $N : P(X) \longrightarrow \mathbb{R}$ by $N(B) = cardB$ is a MV-pseudo norm.

Theorem 2.3. *Let N_1 and N_2 be MV-pseudo norms on A and $\alpha \geq 0$, then*
(i) the function $N : A \longrightarrow \mathbb{R}$, defined by $N(x) = \alpha N_1(x) + N_2(x)$, is an MV-pseudo norm. Moreover, N is an MV-norm, if N_1 and N_2 are MV-norms.
(ii) the map $N(x) = inf\{N_1(z) : z \in \frac{x}{I}\}$ is an MV-pseudo norm, where I is an ideal in A.

THEOREM 2.4. *Let I be an ideal in an MV-algebra A, and N be an MV-pseudo norm on it. Then,*
(i) *the map $n : \frac{A}{I} \longrightarrow \mathbb{R}$ defined by $n(\frac{x}{I}) = \inf\{N(z) : z \in \frac{x}{I}\}$ is an MV-pseudo norm on $\frac{A}{I}$ morovere if for every $x \in A$, $\min \frac{x}{I}$ exists and N is an MV-norm on A, then $n(\frac{x}{I})$ is also an MV-norm on $\frac{A}{I}$.*

If F is filter, similar to the Theorem **??**, $n(\frac{x}{F})$ is also an MV-pseudo norm on $\frac{A}{F}$.

THEOREM 2.5. *Let I be an ideal in an MV-algebra A. Then,*
(i) *the set $I_N = \{x \in A : N(x) = 0\}$ is an ideal in A if N is an MV-pseudo norm on A; moreover if n is an MV-pseudo norm on $\frac{A}{I}$, then $N(x) = n(\frac{x}{I})$ is an MV-pseudo norm on A. Moreover, n is an MV-norm on $\frac{A}{I}$ if and only if $I = I_N$.*

THEOREM 2.6. *Let f be an isomorphism from an MV-algebra $(A_1, \oplus, 0)$ to an MV-algebra $(A_2, \oplus, 0)$. If N_{A_1} is an MV-pseudo norm on A_1, then $N_{A_2} : A_2 \longrightarrow \mathbb{R}$, defined by $N_{A_2}(y) = N_{A_1} \circ f^{-1}(y)$ for every $y \in A_2$, is an MV-pseudo norm on A_2, and $N_{A_2}(f(x)) = N_{A_1}(x)$.*

THEOREM 2.7. *Let A_1 and A_2 be MV-algebras, and N_{A_1} be an MV-pseudo norm on A_1. If $f : A_1 \longrightarrow A_2$ is an epimorphism, then $N_{A_2} : A_2 \longrightarrow \mathbb{R}$ defined by $y \longmapsto \inf\{N_{A_1}(z) : f(z) = y\}$ is an MV-pseudo norm on A_2, and $N_{A_2}(f(x)) \leq N_{A_1}(x)$.*

3. MV-pseudo metrics on MV-algebras

DEFINITION 3.1. A pseudo metric d on an MV-algebra A is called an *MV-pseudo metric* if for every $x, y, a, b \in A$,
$(D5)$ $d(x \oplus y, a \oplus b) \leq d(x, a) + d(y, b)$, and
$(D6)$ $d(x^*, y^*) \leq d(x, y)$.
An *MV-metric* on A is an MV-pseudo metric that satisfies $d(x, y) = 0 \Longleftrightarrow x = y$.

THEOREM 3.2. *If N is an MV-pseudo norm on an MV-algebra A, then $d_N(x, y) = N(x \ominus y) + N(y \ominus x)$ is an MV-pseudo metric on A.*

COROLLARY 3.3. *MV-pseudo metric d_N of Theorem **??**, satisfies the following properties.*
(i) *For every x, $d_N(0, x) + d_N(1, x) = N(1)$, (ii) The mapping d_N is an MV-metric if and only if N is an MV-norm, (iii) For every x, $d_N(x, x^*) \leq N(1)$.*

Remark. From now on, if N is an MV-pseudo norm on an MV-algebra, then d_N is the MV-pseudo metric induced by N in Theorem **??**.
Let A be an MV-algebra and $\mathcal{U}$ be a uniformity on A. By Definition uniformly continuous,
(i) the operation $\oplus : (A \times A, \mathcal{U} \times \mathcal{U}) \to (A, \mathcal{U})$ is *uniformly continuous* if for every $W \in \mathcal{U}$, there exist $U, V \in \mathcal{U}$ such that $U \oplus V \subseteq W$ or equivalently, for every $(x, x') \in U$ and $(y, y') \in V$, $(x \oplus y, x' \oplus y') \in W$;
(ii) the map $* : (A, \mathcal{U}) \to (A, \mathcal{U})$ is *uniformly continuous* if for every $W \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that if $(x, y) \in V$, then $(*(x), *(y)) \in W$.
The pair $(A, \mathcal{U})$ is called a *uniform MV-algebra* if $\oplus$ and $*$ are uniformly continuous.

Let d be an MV-pseudo metric on an MV-algebra A. Then, it is easy to prove that the set $\mathcal{B} = \{U_\epsilon : \varepsilon > 0\}$ is a base for a uniformity $\mathcal{U}_d$ on A, where $U_\epsilon = \{(x, y) : d(x, y) < \epsilon\}$. Thus, by Definition uniformly continuos and $(D5)$ and $(D6)$, the operations $\oplus$ and $*$ are uniformly continuous.

A subset S of an MV-algebra A is said to be *convex* if for any $x, y, z \in A$, $x \leq z \leq y$, and $x, y \in S$ imply that $z \in S$.

Proposition 3.4. *Let A be an MV-algebra, $S \subseteq A$ and $\widehat{S} = \{x \in A : \exists\ y \in S$ such that $x \leq y\}$. Then,*
(i) if $0 \in S$, then S is convex if and only if for any $x, y \in A$, if $x \leq y$ and $y \in S$, then $x \in S$;
(ii) $0 \in \widehat{S}$ and $\widehat{S}$ is the smallest convex set of A containing S;
(iii) if $S \subseteq T$, then $\widehat{S} \subseteq \widehat{T}$;
(iv) $\widehat{S} \oplus \widehat{T} \subseteq \widehat{S \oplus T}$.

Remark. Let d be a pseudo metric on MV-algebra A. We denote the set $\{x : d(x, 0) < r\}$ by $B(r)$ i.e $B(r) = \{x : d(x, 0) < r\}$. Also, we recall that the first part of the proof of the following theorem is from [**?**].

Theorem 3.5. *Let $\{U_n\}_{n \geqslant 0}$ be a family of subsets of an MV-algebra A such that $0 \in U_n$ and $U_{n+1} \oplus U_{n+1} \subseteq U_n$ for any $n \geq 0$. Then there is an MV-pseudo metric d on A such that the operations $\oplus$ and $*$ are uniformly continuous on $(A, \mathcal{U}_d)$ and for any $n \geq 0$,*

$$\{x : d(x, 0) < 1/2^n\} \subseteq \widehat{U_n} \subseteq \{x : d(x, 0) < 2/2^n\}.$$

Moreover, d is an MV-metric if and only if $\bigcap_{n \geq 0} \widehat{U}_n = 0$.

Proof. Let $V(1) = U_0$, $n \geq 0$ and assume that $V(\frac{m}{2^n})$ are defined for each $m = 1, 2, 3, ..., 2^n$ such that $0 \in V(\frac{m}{2^n})$. Put then $V(\frac{1}{2^{n+1}}) = U_{n+1}$, $V(\frac{2m}{2^{n+1}}) = V(\frac{m}{2^n})$ for $m = 1, 2, 3, ..., 2^n$ and for each $m = 1, 2, 3, ..., 2^n - 1$, $V(\frac{2m+1}{2^{n+1}}) = V(\frac{m}{2^n}) \oplus U_{n+1} = V(\frac{m}{2^n}) \oplus V(\frac{1}{2^{n+1}})$. We also define $V(\frac{m}{2^n}) = A$, when $m > 2^n$. By induction on n we prove that for any $m > 0$ and $n \geq 0$,

$$(*) \quad V(\frac{m}{2^n}) \oplus V(\frac{1}{2^n}) \subseteq V(\frac{m+1}{2^n}).$$

First notice that if $m + 1 > 2^n$, then $(*)$ is obviously true. Let $m < 2^n$. If $n = 1$, then m is also 1, so $V(\frac{1}{2}) \oplus V(\frac{1}{2}) = U_1 \oplus U_1 \subseteq U_0 = V(1)$. Asume that $(*)$ holds for some n. We verify it for $n + 1$. If $m = 2k$, then by the definition of $V(\frac{2m+1}{2^{n+1}})$, $V(\frac{m}{2^{n+1}}) \oplus V(\frac{1}{2^{n+1}}) = V(\frac{2k}{2^{n+1}}) \oplus V(\frac{1}{2^{n+1}}) = V(\frac{k}{2^n}) \oplus V(\frac{1}{2^{n+1}}) = V(\frac{2k+1}{2^n})$. Suppose now that $m = 2k + 1 < 2^{n+1}$ for some $n \geq 0$. Then
$V(\frac{m}{2^{n+1}}) \oplus V(\frac{1}{2^{n+1}}) = V(\frac{2k+1}{2^{n+1}}) \oplus U_{n+1} = V(\frac{k}{2^n}) \oplus U_{n+1} \oplus U_{n+1} \subseteq V(\frac{k}{2^n}) \oplus U_n = V(\frac{k}{2^n}) \oplus V(\frac{1}{2^n})$. But by the inductive assumption, $V(\frac{m}{2^{n+1}}) \oplus V(\frac{1}{2^{n+1}}) \subseteq V(\frac{k+1}{2^n}) = V(\frac{m+1}{2^{n+1}})$. By Proposition **??**, for any $r \geq 0$, $\widehat{V(r)}$ is a convex set containing 0, it is easy to derive that the map $f : A \longrightarrow \mathbb{R}$ defined by $f(x) = inf\{r : x \in \widehat{V(r)}\}$ is increasing bounded function. Define the map $N : A \longrightarrow \mathbb{R}$ by $N(x) = sup\{f(x \oplus z) - f(z) : z \in A\}$. The function N is obviously well defined and increasing. In a similar method with the proof of Theorem **??**, we can show that $d_N(x, y) = N(x \ominus y) + N(y \ominus x)$ is an MV-pseudo metric. By $(D5)$ and $(D6)$, we can prove that the operations $\oplus$ and $*$ are uniformly continuous on $(A, \mathcal{U}_{d_N})$. Let us prove that d_N satisfies

$$\{x : d_N(x, 0) < \frac{1}{2^n}\} \subseteq \widehat{U_n} \subseteq \{x : d_N(x, 0) \leq \frac{2}{2^n}\}.$$

Notice that $f(0) = 0$, hence if $d_N(x, 0) < \frac{1}{2^n}$, then $f(x) = f(x \oplus 0) - f(0) \leq N(x) = d_N(x, 0) < \frac{1}{2^n}$. Hence for some $0 \leq r < \frac{1}{2^n}$, $x \in \widehat{V(r)}$. Since $V(r) \subseteq V(\frac{1}{2^n}) = U_n$, $x \in \widehat{V(r)} \subseteq \widehat{V(\frac{1}{2^n})} = \widehat{U_n}$. Now let $x \in \widehat{U_n}$. Then there is a $x' \in U_n$ such that $x \leq x'$.

Clearly for any $z \in A$, there exists a $k \geq 0$ such that $\frac{k-1}{2^n} \leq f(z) \leq \frac{k}{2^n}$. Since $z \in \widehat{V(\frac{k}{2^n})}$, there is a $z' \in V(\frac{k}{2^n})$ such that $z \leq z'$. From condition $(*)$ it follows that $z' \oplus x' \in V(\frac{k}{2^n}) \oplus V(\frac{1}{2^n}) \subseteq V(\frac{k+1}{2^n})$ and from $z \oplus x \leq z' \oplus x'$ deduces that $z \oplus x \in \widehat{V(\frac{k+1}{2^n})}$. Hence $f(x \oplus z) - f(z) \leq \frac{k+1}{2^n} - \frac{k-1}{2^n} = \frac{2}{2^n}$.

In the end of proof, let us prove that d_N is an MV-metric if and only if $\bigcap_{n\geq 0} \widehat{U_n} = 0$. Let $\bigcap_{n\geq 0} \widehat{U_n} = 0$ and $d_N(x,y) = 0$. Then $N(x \ominus y) = N(y \ominus x) = 0$. Hence for any $n \geq 0$, $x \ominus y$ and $y \ominus x$ are in $\widehat{U_n}$. This concludes that $x \ominus y = y \ominus x = 0$ and so $x = y$. Therefore d_N is metric.

Conversely let d_N be metric and $x \in \bigcap_{n\geq 0} \widehat{U_n}$. Since $\widehat{U_n} \subseteq \{x : d_N(x,0) \leq \frac{2}{2^n}\}$ for every $n \geq 0$, we derive that $d_N(x,0) = 0$. This implies that $x = 0$. □

THEOREM 3.6. *Let A be a MV-algebra. Then, there is an MV-pseudo metric d on A such that $(A,\mathcal{U}_d)$ is a uniform MV-algebra if and only if there is a topology τ on A such that (A,τ) is a topological MV-algebra and τ hsa a countable local base at 0. Moreover, d is continuous in (A,τ).*

PROPOSITION 3.7. *Let $S = \{N_i : i \in I\}$ be a chain of MV-pseudo norms on an MV-algebra A. Then, there exists a uniformity $\mathcal{U}$ on A such that $(A,\mathcal{U})$ is a uniform MV-algebra.*

PROPOSITION 3.8. *Suppose A is an MV-algebra, I is an ideal and $q : A \longrightarrow \frac{A}{I}$, given by $q(x) = \frac{x}{I}$, is the quotient map. Then there are uniformities η_I and ε_I on A and $\frac{A}{I}$ such that (A,η_I) and $(\frac{A}{I},\varepsilon_I)$ are uniform MV-algebras and $q : (A,\eta_I) \to (\frac{A}{I},\varepsilon_I)$ is uniformly continuous.*

PROPOSITION 3.9. *Let N be an MV-pseudo norm and I be an ideal in an MV-algebra A. Then there exists an MV-pseudo metric D_n on $\frac{A}{I}$ such that $(\frac{A}{I},\mathcal{U}_{D_n})$ is a uniform MV-algebra and the quotient map $q : (A,\mathcal{U}_{d_N}) \longrightarrow (\frac{A}{I},\mathcal{U}_{D_n})$, given by $q(x) = \frac{x}{I}$, is uniformly continuous.*

4. Conclusion

In this article is to introduce MV-pseudo norms, MV-pseudo metric and MV-metric and its relation to uniform continuity are discussed.

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