On pseudoconvexity conditions and static spacetimes

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Abstract. Recently, the relationship between (geodesics) convexity, connectedness, and completeness properties in Riemannian manifolds $(\Sigma; h)$ and the causal properties in Lorentzian static spacetimes $(M; g) = (\mathcal{R} \times \Sigma; dt^2 + h)$ is studied. In this paper, some sufficient conditions are introduced to $(\Sigma; h)$ be geodesically convex.

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1. Introduction

Let $(\Sigma; h)$ be a Riemannian manifold. The product manifold $M = \mathcal{R} \times \Sigma$ endowed with the direct sum metric $g = dt^2 + h$ is a Lorentzian manifold $(M; g)$ which encodes all the information of $(\Sigma; h)$. For instance, geodesics on M project to geodesics on Σ, and every geodesic on Σ comes from such a projection. Namely, every null geodesic γ of $(M; g)$ is of the form $t \longrightarrow (t, (t))$ where η is a h-arclength geodesic. In fact, Lorentzian geometry is richer than Riemannian geometry and the converse inclusion does not hold since Riemannian manifolds do not encode any cone dynamics, i.e. any causality theory. In general relativity, a *spacetime* is a pair (M, g) where M is a real, connected, C^∞ Hausdorff manifold of dimension two or more, and g is a globally defined C^∞ Lorentzian metric on M of signature $(+, -, ..., -)$. When there is no ambiguity, we use M to refer to the spacetime (M, g). We say that a vector $v \in T_pM$ is *timelike* if $g_p(v, v) > 0$, *causal* if $g_p(v, v) \geq 0$, *null* if $g_p(v, v) = 0$ and *spacelike* if $g_p(v, v) < 0$. A smooth curve is called *future directed timelike curve* if its tangent vector is everywhere timelike future pointing vector and similarly for spacelike, causal, null future directed (or null past directed) curve can be defined. If $p, q \in M$, then q is in the *chronological future of* p, written $q \in I^+(p)$ or $p \prec q$, if there is a timelike future pointing curve $\gamma : [0, 1] \to M$ with $\gamma(0) = p$, and $\gamma(1) = q$; similarly, q is in the *causal future of* p, written $q \in J^+(p)$ or $p \preceq q$, if there is a future pointing causal curve from p to q. For any point, p, the set $I^+(p)$ is open; but $J^+(p)$ need not, in general, be closed. $J^+(p)$ is always a subset of the closure of

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$I^+(p)$. A spacetime M causal if it has no point p with a non-degenerate causal curve that starts and ends at p. A spacetime M is said to be distinguishing if for all points p and q in M, either $I^+(p) = I^+(q)$ or $I^-(p) = I^-(q)$ implies $p = q$. If each point p has arbitrarily small neighborhoods in which any causal curve intersects in a single component, M satisfies the condition of *strong causality*. A distinguishing spacetime M is said to be *causally continuous* at p if the set-valued functions I^+ and I^- are both inner continuous and outer continuous at p. The set-valued function $I^\pm$ is said to be *inner continuous* at $p \in M$ if for each compact set $K \subseteq I^\pm(p)$, there exists a neighborhood $U(p)$ of p such that $K \subseteq I^\pm(q)$ for each $q \in U(p)$. The set-valued function $I^\pm$ is *outer continuous* at p if for each compact set K in the exterior of $\overline{I^\pm(p)}$ there exists some neighborhood $U(p)$ of p such that for each $q \in U(p)$, K is in the exterior of $\overline{I^\pm(q)}$. We recall that $I^\pm$ is always inner continuous. If M is causal and $J^\pm(p)$ is closed for all $p \in M$, then M is *causally simple*. A spacetime M is said to be *globally hyperbolic* if M is strongly causal and $J^+(p) \cap J^-(q)$ is compact for all p and q in M. A spacetime $(M; g)$ is called (causal, null or maximally null) pseudoconvex, if for any compact set K, there exists another compact set K^*, such that each geodesic of the respective type with both endpoints in K must be entirely contained in K^*. Clearly pseudoconvexity implies causal pseudoconvexity which implies null pseudoconvexity which again is stronger than maximal null pseudoconvexity. Riemannian version of this is defined similally. A Riemannian manifold $(\Sigma; h)$ is called (minimally) pseudoconvex, if for any compact set C there exists another compact set C^*, such that each (minimal) geodesic with endpoints in C must be entirely contained in C^*. Clearly, pseudoconvexity implies minimal pseudoconvexity (see [3, 5]).

Definition 1.1. Assume $p_n \to p$ and $q_n \to q$ for distinct points p and q in a spacetime M. We say that the spacetime M has the limit geodesic segment property (LGS), if each pair p_n and q_n can be joined by a "geodesic segment", then there is a limit geodesic segment from p to q. Namely, for every sequence of geodesics γ_n from p_n to q_n where $p_n \to p, q_n \to q$, there are a subsequence γ_k and a geodesic segment from p to q such that γ_k converges h-uniformly to γ. Similarly, causal, null, and maximal null LGS property can be defined by restricting the condition "geodesic segment" to causal, null, and maximal null geodesics, respectively.

2. Main results

In Ref. [5], for the first time, it is shown that the pseudoconvexity and LGS property are equivalent.

Proposition 2.1. *[5, Proposition 4] Let (M, g) be a strongly causal spacetime. Then, it is (null or maximal null) causal pseudoconvex if and only if it has the (null or maximal null) causal LGS property.*

Also, a Riemannian manifold $(\Sigma; h)$ is disprisoning if no inextensible geodesic $\gamma : [0, b) \to \Sigma$ imprisons in a compact set. In Ref. [3], a Riemannian version of Proposition 2.1 is proved.

Lemma 2.2. *A Riemannian manifold $(\Sigma; h)$ satisfies the LGS if and only if it is disprisoning and pseudoconvex.*

By Hopf-Rinow theorem, For any connected Riemannian manifold Σ, if exp_p is defined on all of $T_p\Sigma$, then any point of Σ can be joined to p by a minimizing geodesic and any two points of Σ can be joined a minimizing geodesic (Σ is geodesically convex) if Σ is geodesically complete (i.e. the domain of geodesics can be extended to all real numbers $\mathcal{R}$) or quivalently every closed and bounded subsets of Σ are compact or quivalently Σ is a complete metric space. The converse of the last statment in the theorem is false, by taking Σ to be an open ball of $\mathcal{R}^n$ as an open submanifold.

THEOREM 2.3. *[2, Theorem 3.67.] and [3, Lemma 3.4.] Let $(\Sigma; h)$ be a Riemannian manifold and $M = \mathcal{R}\times\Sigma$ be the Lorentzian manifold with the direct sum metric $g = dt^2+h$. The following statments are hold:*

1) *$(M; g)$ is globally hyperbolic if and only if $(\Sigma; h)$ is geodesically complete.*
2) *$(M; g)$ is causally simple if and only if $(\Sigma; h)$ is geodesically convex.*

There are a problem that say "the pseudoconvexity of Σ implies the geodesically convexity of Σ". This leads to solve a conjecture that say "the nul pseudoconvexity of M implies the geodesically convexity of M" (see [6]). Also, it is shown that if $(\Sigma; h)$ is a pseudoconvex Riemannian manifold which admits an equidimensional embedding into a complete manifold then Σ is geodesically convex.

PROPOSITION 2.4. *Let $(\Sigma; h)$ be a Riemannian manifold. Any limit curve of a sequence of minimal geodesics in Σ is a minimal geodesic.*

PROOF. Let $(\Sigma; h)$ be a Riemannian manifold and $M = \mathcal{R} \times \Sigma$ be the Lorentzian manifold with the direct sum metric $g = dt^2+h$ and σ_n be a sequence of minimal geodesics in Σ converges to σ. Set $\gamma_n := (b_n t, \sigma_n)$ and $\gamma := (bt, \sigma)$ where $b_n = L^h(\sigma_n)$ and $b = L^h(\sigma)$. It is clear that γ_n is a sequence of maximal null geodesics converging to γ in M. By [6, Remark 2], γ is a maximal null geodesic. Finally, [3, Lemma 3.1] implies σ is minimal geodesic. □

PROPOSITION 2.5. *If $(\Sigma; h)$ is a disprisoning and pseudoconvex (LGS) Riemannian manifold without conjugate points, then Σ is geodesically convex.*

PROOF. By [4, Theorem 2.2], it concludes that Σ is geodesically connected. Now, let $M = \mathcal{R} \times \Sigma$ be the Lorentzian manifold with the direct sum metric $g = dt^2 + h$. According to the hypothesis, M is a disprisoning and pseudoconvex space-time that is also geodesically connected. So, By [1, Proposition 2], (M, g) is causally simple and Theorem 2.3 implies that $(\Sigma; h)$ is geodesically convex. □

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