Characterization of a special case of hom-Lie superalgebra

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Abstract. In this paper, we introduce the notion of sympathetic hom-Lie superalgebras. We prove some results on sympathetic multiplicative hom-Lie superalgebras with surjective α. In particular, we find some equivalence condition in which a sympathetic graded hom-ideal is direct factor of multiplicative hom-Lie superalgebra.

Keywords: hom-Lie superalgebra, Sympathetic hom-Lie superalgebra, multiplicative hom-Lie superalgebra

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1. Introduction

The notion of a hom-Lie algebra was introduced by Hartwig et al. [7] as part of study of deformations of the Witt and the Virasoro algebras, which are widely utilized in the theoretical physics; such as string theory, vertex models in conformal field theory, quantum mechanics and quantum field theory. In a hom-Lie algebra, the jacobi identity is twisted by a linear map, called the hom-Jacobi identity. Many results of hom-Lie algebras were generalized to hom-Lie superalgebras by authors [1, 2, 4, 5]. The authors introduced the complete Lie superalgebra and recently the notion of compact hom-Lie superalgebra was introduced in [2, 4] that the sympathetic hom-Lie superalgebra is a special case of it. Throughout this paper we fix a ground field $\mathbb{K}$, which is algebraically closed of characteristic zero. All $\mathbb{Z}_2$-graded vector spaces are considered over $\mathbb{K}$ and linear maps are $\mathbb{K}$-linear maps. Each element in the hom-Lie superalgebra is supposed to be homogeneous and degree of x is denoted by $|x|$.

2. Main Results

The main topics covered in this article are hom-Lie superalgebras. So, at the first, hom-Lie superalgebras and some of their related definitions are presented.

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The concept of a hom-Lie superalgebra was introduced by Ammar and Makhlouf in [1]. At first let us recall some basic concepts from [1, 8].

DEFINITION 2.1. [8] A Lie superalgebra is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, together with a graded Lie bracket $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ of degree zero, i.e. $[.,.]$ is a bilinear map with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, such that for homogeneous elements $x, y, z \in \mathfrak{g}$, the following identities hold:

- $[x, y] = -(-1)^{|x||y|}[y, x]$,
- $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$.

DEFINITION 2.2. [1] A hom-Lie superalgebra is a triple $(\mathfrak{g}, [.,.], \alpha)$ consisting of a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, an even linear map (bracket) $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and an even homomorphism $\alpha : \mathfrak{g} \to \mathfrak{g}$ satisfying the following supersymmetry and hom-Jacobi identity, i.e.

- $[x, y] = -(-1)^{|x||y|}[y, x]$,
- $(-1)^{|x||z|}[\alpha(x), [y, z]] + (-1)^{|y||x|}[\alpha(y), [z, x]] + (-1)^{|z||y|}[\alpha(z), [x, y]] = 0$,

where x, y and z are homogeneous elements in $\mathfrak{g}$. If $|x|$ appears in some expression in this paper, we always regard x as a Z_2-homogeneous element and $|x|$ as its Z_2-degree.

- A hom-Lie superalgebra is called *multiplicative* hom-Lie superalgebra if α is an algebraic morphism, i.e. for any $x, y \in \mathfrak{g}$ we have
  $\alpha([x, y]) = [\alpha(x), \alpha(y)]$.
- A hom-Lie superalgebra is called *regular* hom-Lie superalgebra, if α is an algebraic automorphism.
- An even homomorphism $f : \mathfrak{g} \to \mathfrak{g}'$ where $(\mathfrak{g}, [.,.], \alpha)$ and $(\mathfrak{g}', [.,.]', \beta)$ are two hom-Lie superalgebras is said to be a homomorphism of hom-Lie superalgebras, if
  $f[u, v] = [f(u), f(v))]'$,
  $f \circ \alpha = \beta \circ f$.

REMARK 2.3. When $\alpha = id$, we get the classical Lie superalgebra.

EXAMPLE 2.4. [1] (Affine hom-Lie superalgebra). Let $V = V_0 \oplus V_1$ be a 3-dimensional superspace where V_0 is generated by e_1, e_2 and V_1 is generated by e_3. The triple $(V, [,], \alpha)$ is a hom-Lie superalgebra defined by $[e_1, e_2] = e_1$, $[e_1, e_3] = [e_2, e_3] = [e_3, e_3] = 0$ and α is any homomorphism.

A graded sub-vector space $A \subseteq \mathfrak{g}$ is a hom-subalgebra of $(\mathfrak{g}, [.,.], \alpha)$, if $\alpha(A) \subseteq A$ and A is closed under the bracket operation $[.,.]$, i.e. $[A, A] \subseteq A$. Also, a graded hom-subalgebra A is called a hom-ideal of $\mathfrak{g}$, denoted by $A \lhd \mathfrak{g}$, if $[A, \mathfrak{g}] \subseteq A$. Moreover, if $[A, A] = 0$, then A is called Abelian.

DEFINITION 2.5. [6] The center of a hom-Lie superalgebra $\mathfrak{g}$, denoted by $C(\mathfrak{g})$, is the set of elements $x \in \mathfrak{g}$ satisfying $[x, \mathfrak{g}] = 0$.

Now, we recall the notion of an α^t-derivation.

DEFINITION 2.6. Let $(\mathfrak{g}, [.,.], \alpha)$ be a multiplicative hom-Lie superalgebra. For any nonnegative integer t, a linear map $D : \mathfrak{g} \to \mathfrak{g}$ of degree d is called an α^t-derivation of the multiplicative hom-Lie superalgebra $(\mathfrak{g}, [.,.], \alpha)$, if

- $[D, \alpha] = 0$, i.e. $D \circ \alpha = \alpha \circ D$,
- $D([a, b]) = [D(a), \alpha^t(b)] + (-1)^{d|a|}[\alpha^t(a), D(b)]$ for all $a, b \in \mathfrak{g}$.

For any $a \in \mathfrak{g}$ satisfying $\alpha(a) = a$, define $ad_t(a) : \mathfrak{g} \to \mathfrak{g}$ by

$$ad_t(a)(b) = [a, \alpha^t(b)],$$

for all $b \in \mathfrak{g}$.

Definition 2.7. A hom-Lie superalgebra $\mathfrak{g}$ is called sympathetic hom-Lie superalgebra if $\mathfrak{g}$ satisfies the following two conditions.

- $C(\mathfrak{g}) = 0$
- $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$
- $Der_{\alpha^{t+1}}(\mathfrak{g}) = ad_t(\mathfrak{g})$,

for any nonnegative integer t.

Definition 2.8. Let $\mathfrak{g}$ be a hom-Lie superalgebra, A be a graded hom-ideal of $\mathfrak{g}$. Then A is said to be a direct factor if there exists a graded hom-ideal B of $\mathfrak{g}$ such that $\mathfrak{g} = A \oplus B$.

Definition 2.9. Let $\mathfrak{g}$ be a hom-Lie superalgebra, A be a graded subspace of $\mathfrak{g}$. Then A is called characteristic hom-ideal if for every $D \in Der_{\alpha^{t+1}}(\mathfrak{g}), D(A) \subseteq A$.

Lemma 2.10. *Let $(\mathfrak{g}, [.,.], \alpha)$ be a multiplicative hom-Lie superalgebra with surjective α and A be a graded hom-ideal of $\mathfrak{g}$. If A is perfect, then A is a characteristic hom-ideal of $\mathfrak{g}$.*

Proof. Let $D \in Der_{\alpha^{t+1}}(\mathfrak{g})$ and $x, y \in A$. Then by defenition of α^t-derivation we have

$$D([x, y]) = [D(x), \alpha^t(y)] + (1)|D||x|[\alpha^s(x), D(y)] \in A.$$

Since A is perfect, then we have $D(A) \subseteq A$. Thus A is a characteristic hom-ideal. ☐

By above notation, we have the following proposition.

Proposition 2.11. *Let $(\mathfrak{g}, [.,.], \alpha)$ be a perfect multiplicative hom-Lie superalgebra with surjective α , A be a graded hom-ideal of $\mathfrak{g}$. If A is a direct factor of $\mathfrak{g}$, then A is perfect.*

Proof. Since A is a direct factor of $\mathfrak{g}$, then there exists a graded hom-ideal B of $\mathfrak{g}$ such that $\mathfrak{g} = A \oplus B$, in particular, $[A, B] = \{0\}$. It follows that $[\mathfrak{g}, \mathfrak{g}] = [A, A] \oplus [B, B]$. So both A and B are perfect. ☐

Proposition 2.12. *Let $(\mathfrak{g}, [.,.], \alpha)$ be a multiplicative hom-Lie superalgebra with surjective α and trivial center. Let A be a direct factor of $\mathfrak{g}$. Then $C(A) = \{0\}$.*

Now we consider the sympathetic hom-Lie superalgebra to state some results.

Proposition 2.13. *Let $(\mathfrak{g}, [.,.], \alpha)$ be a multiplicative hom-Lie superalgebra with surjective α and A be a sympathetic graded hom-ideal of $\mathfrak{g}$. Then there exists a graded hom-ideal B such that $\mathfrak{g} = A \oplus B$.*

Proof. Let $B = C_{\mathfrak{g}}(A)$. Since α is surjective then $C_{\mathfrak{g}}(A)$ is a graded hom-ideal of $\mathfrak{g}$. We know that $A \lhd \mathfrak{g}$, then For any $x \in \mathfrak{g}$, $ad_t(x) \in Der_{\alpha^{t+1}}(A)$. By using $Der_{\alpha^{t+1}}(A) = ad_t(A)$, there exists a derivation D in $Der_{\alpha^{t+1}}(A)$ such that $ad_t(x) = D$. So there exists $y \in A$ such that

$$D(z) = [x, \alpha(z)] = [y, \alpha(z)],$$

for all $z \in A$. Then $[x - y, \alpha(z)] = 0$ and then $x - y \in C_{\mathfrak{g}}(A) = B$. Hence $x = b + y$ for some b $\in B$. $A \cap B = A \cap C_{\mathfrak{g}}(A) = C_A(A) = \{0\}$, since A is sympathetic. Therefore $\mathfrak{g} = A \oplus B$. □

PROPOSITION 2.14. *Let* $(\mathfrak{g}, [.,.], \alpha)$ *be a sympathetic multiplicative hom-Lie superalgebra with surjective* α *and* A *be a graded hom-ideal of* $\mathfrak{g}$*. Then* A *is a direct factor of* $\mathfrak{g}$ *if and only if* A *is sympathetic.*

By using above Proposition, we have the following consequences immediately.

COROLLARY 2.15. *Let* $(\mathfrak{g}, [.,.], \alpha)$ *be a multiplicative hom-Lie superalgebra with surjective* α *and* A *be a sympathetic graded hom-ideal of* $\mathfrak{g}$*. Then* A *is a direct factor of* $\mathfrak{g}$*.*

COROLLARY 2.16. *Let* $(\mathfrak{g}, [.,.], \alpha)$ *be a multiplicative hom-Lie superalgebra with surjective* α *and* A *be a sympathetic graded hom-ideal of* $\mathfrak{g}$*. If* $Der_{\alpha^{t+1}}(\mathfrak{g}) = ad_t(\mathfrak{g})$*, then* $Der_{\alpha^{t+1}}(A) = ad_t(A)$*.*

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