Transitivity in IFS over arbitrary shift spaces

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Abstract. The orbit of a point $x \in X$ in a classical iterated function system (IFS) is defined as $\{f_u(x) = f_{u_n} \circ \cdots \circ f_{u_1}(x) : u = u_1 \cdots u_n$ is a word of a full shift on finite symbols$\}$. In other words, an IFS is parameterized over the full shift. Here, we parameterize our IFS over an arbitrary shift space Σ. In particular, we associate to $\sigma \in \Sigma$ a non-autonomous system $(X,\ f_\sigma)$ where trajectory of $x \in X$ is defined as $x,\ f_{\sigma_1}(x),\ f_{\sigma_1\sigma_2}(x), \ldots$. We show that for a transitive IFS and a sofic Σ, there is a transitive $t \in \Sigma$ such that the non-autonomous system $(X,\ f_t)$ is transitive. This is not true for the case where Σ is non-sofic.

Keywords: iterated function systems (IFS), non-autonomous system, transitivity.

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1. Introduction

In a classical dynamical system, here called *conventional dynamical system*, we have a phase space and a unique map where the trajectories of points are obtained by iterating this map. However, in various problems, including applied ones, one may have some finite sequence of maps in place of a single map acting on the same phase space. For instance, in Physics by two or more maps have appeared in [**1**, **11**], Economy in [**13**] and Biology in [**4**]. In Mathematics, this has been studied either by non-autonomous systems in many literature such as [**9**] or as iterated function system (IFS) for constructing and studying some fractals in [**5**, **8**] or for investigating dynamical properties in many places such as [**2**, **3**, **6**, **7**].

In a "classical" IFS, a compact metric space X and a set of some k finite continuous functions $\{f_0, \cdots, f_{k-1}\}$ on X are assumed and the trajectory of a point $x \in X$ is considered to be the action on x of the sequence of freely combination of those maps, or action on x of combination of those maps over the words of a full shift: just write

$$f_u = f_{u_1} \circ \cdots \circ f_{u_m} \tag{1}$$

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where $u = u_1 \cdots u_m$ is a word of the full shift over k symbols. However, in some physical problems, such freely action is not possible. In other words, there are some words that one cannot perform (1). This is the case where a subshift instead of the full shift must be considered and it is of our interest.

2. Preliminaries

2.1. Iterated function systems. Throughout the paper, X will be a compact metric space. The *classical* iterated function system (IFS) consists of finitely many continuous self maps $\mathcal{F} = \{f_0, \ldots, f_{k-1}\}$ on X. The *forward orbit* of a point $x \in X$, denoted by $\mathcal{O}^+(x)$, is the set of all values of finite possible combinations of f_i's at x. We need the following equivalent statement: Let Σ_F be the full shift on k symbols and let $\mathcal{L}(\Sigma_F)$ called the *language of* Σ_F be the set of words or blocks. Define $f_u(x)$ as in (1) and set $\mathcal{O}^+(x) := \{f_u(x) : \ u \in \mathcal{L}(\Sigma_F)\}$. Such iterated function systems, here called *classical IFS*, have been the subject of study for quite a long time.

Here we define an IFS to be

$$\mathfrak{I} = (X, \mathcal{F} = \{f_0, \ldots, f_{k-1}\}, \Sigma). \tag{2}$$

where f_i is continuous and Σ is an arbitrary subshift on k symbols, not necessarily the full shift Σ_F as in the classical IFS. Hence we use a good deal of symbolic dynamics, see for instance [10] for this subject. By this setting, Σ_F above will be replaced with Σ and thus $\mathcal{O}^+(x) = \{f_u(x) : \ u \in \mathcal{L}(\Sigma)\}$ is the forward orbit of x. In particular, $f_u(f_v(x)) = f_{vu}(x)$ whenever vu is admissible or equivalently $vu \in \mathcal{L}(\Sigma)$. Let $u = u_1 \cdots u_n \in \mathcal{L}(\Sigma)$ and set $u^{-1} := u_n \cdots u_1$. Then for $A \subseteq X$,

$$\begin{aligned} (f_u)^{-1}(A) &= (f_{u_n} \circ \cdots \circ f_{u_1})^{-1}(A) \\ &= f_{u_1}^{-1} \circ \cdots \circ f_{u_n}^{-1}(A) \\ &= f_{u^{-1}}^{-1}(A), \end{aligned}$$

where for the last equality, we used (1). Also

$$\begin{aligned} f_{u^{-1}}^{-1}(f_{v^{-1}}^{-1}(A)) &= f_{v^{-1}u^{-1}}^{-1}(A) = f_{(uv)^{-1}}^{-1}(A) \\ &= (f_{uv})^{-1}(A). \end{aligned}$$

Thus the backward orbit and the (full) orbit of a point $x \in X$ are $\mathcal{O}_-(x) = \{f_{u^{-1}}^{-1}(x) : u \in \mathcal{L}(\Sigma)\}$ and $\mathcal{O}(x) = \mathcal{O}_-^+(x) = \mathcal{O}^+(x) \cup \mathcal{O}_-(x)$ respectively.

When all f_i's are homeomorphisms, the backward, forward and full trajectory of x is defined.

We say $\mathcal{F} = \{f_0, \ldots, f_{k-1}\}$ in $(X, \mathcal{F} = \{f_0, \ldots, f_{k-1}\}, \Sigma)$ is *surjective*, if all f_i's are surjective.

When $k = 1$ and $\Sigma = \{0^\infty\}$, we simply have the classical dynamical system, here called *conventional dynamical system* denoted either by the pair (X, f) or $\mathfrak{I} = (X, \{f_0\}, \{0^\infty\})$. By the above setting, the following definition looks natural.

DEFINITION 2.1. *Consider $\mathfrak{I}$ as in (2) and let U and V be arbitrary open sets in X. Then $\mathfrak{I}$ is* forward transitive, *if there is $x \in X$ such that $\{f_u(x) : \ u \in \mathcal{L}(\Sigma)\}$ is dense in X.* Backward transitivity *and* transitivity *is likewise defined.*

DEFINITION 2.2. *Let $\mathfrak{I} = (X, \mathcal{F}, \Sigma)$ be an IFS. Then $\mathfrak{I}$ is called* (forward) transitive along an orbit $\sigma \in \Sigma$, *if the non-autonomous system (X, f_σ) is (forward) transitive.*

3. Transitivity in IFS vs transitivity in the subshift

Transitivity in dynamical systems is a sort of richness in dynamics. Thus when an IFS has transitivity along an orbit, we in fact have a non-autonomous transitive system. Recall that transitivity along an orbit defined in Definition 2.2 implies the transitivity of the system defined in Definition 2.1 and the converse is not necessarily true. Having these in mind, we like to address the following questions.

(1) Is there any sufficient condition on Σ such that two notions of transitivity in Definition 2.1 and Definition 2.2 coincides?
(2) In which situation there is a transitive $t \in \Sigma$ such that for some $x \in X$, $\overline{O_t^+(x)} = X$?
(3) How large is the set

$$S = S(\mathfrak{I}) := \{\sigma \in \Sigma : \exists x \in X \text{ s.t. } \overline{\mathcal{O}_\sigma^+(x)} = X\}? \tag{3}$$

The following relatively similar propositions are stated due to the different types of transitivity given in Definitions 2.1 and 2.2.

Proposition 3.1. *Let $(X, \mathcal{F} = \{f_0, \dots, f_{k-1}\}, \Sigma)$ be an IFS with $\mathcal{F}$ surjective. Then the following are equivalent.*

(1) *For some $x \in X$, $\overline{\mathcal{O}^+(x)} = X$.*
(2) *Whenever E is a closed subset of X and for any $i \in \mathcal{A}$, $E \subseteq f_i^{-1}E$, then either $E = X$ or E is nowhere dense. (Equivalently, whenever U is an open subset of X and for any $i \in \mathcal{A}$, $f_i^{-1}U \subseteq U$, then $U = \emptyset$ or U is dense.)*
(3) *Whenever W, V are non-empty open sets, then there is $u \in \mathcal{L}(\Sigma)$ such that $(f_u)^{-1}V \cap W \neq \emptyset$.*
(4) *The set $\{x \in X | \overline{\mathcal{O}^+(x)} = X\}$ is residual in X.*

Although the following proposition, a very classical result in conventional dynamical systems, has been declared for the case when subshift is over an alphabet with finite characters, the proof (not presented here) is valid for a subshift over infinite characters and thus for a general non-autonomous system as well. For non-autonomous systems, Yan, Zeng and Zang in [12] have a similar result for the special case where their non-autonomous system $\{f_{\sigma_i}\}_{i=1}^\infty$ on a compact metric space (X, d) is uniformly convergent to a map f and besides $d(f_{\sigma_n \cdots \sigma_{2n-1}}, f^n) \to 0$ as $n \to \infty$.

Proposition 3.2. *Let $(X, \mathcal{F} = \{f_0, \dots, f_{k-1}\}, \Sigma)$ be an IFS with $\mathcal{F}$ surjective and $\sigma = \sigma_1\sigma_2 \cdots \in \Sigma$. Then the following are equivalent.*

(1) *For some $x \in X$, $\overline{O_\sigma^+(x)} = X$.*
(2) *Assume E is a closed subset of X and for some j, $f_{\sigma_1 \cdots \sigma_j}^{-1} E \subset f_{\sigma_1 \cdots \sigma_j u}^{-1} E$ whenever $\sigma_1 \cdots \sigma_j u \subset \sigma$, then either $E = X$ or E is nowhere dense. (Equivalently, suppose U is an open subset and for some j, $f_{\sigma_1 \cdots \sigma_j u}^{-1} U \subset f_{\sigma_1 \cdots \sigma_j}^{-1} U$ whenever $\sigma_1 \cdots \sigma_j u \subset \sigma$, then $U = \emptyset$ or U is dense.)*
(3) *Whenever W, V are non-empty open sets, then there is $n \in \mathbb{N}$ such that*

$$(f_{\sigma_1 \cdots \sigma_n})^{-1} W \cap V \neq \emptyset. \tag{4}$$

(4) *The set $\{x \in X | \overline{\mathcal{O}_\sigma^+(x)} = X\}$ is residual in X.*

In our next proposition we will show that when Σ is an irreducible sofic and the respective IFS is transitive, then for some $x \in X$, there is a transitive $\sigma \in \Sigma$ such that the transitivity of the IFS occurs along this transitive σ. This will give an answer to questions

1 and 2 on the beginning of this section for special cases where Σ is an irreducible sofic. First we recall a classical result.

Proposition 3.3. *Let $\mathfrak{I} = (X, \mathcal{F}, \Sigma)$ be an IFS, $\mathcal{F}$ surjective and Σ an irreducible sofic. Then, $\mathfrak{I}$ is forward transitive iff there is a forward transitive $t \in \Sigma$ and some $x \in X$ such that $\overline{\mathcal{O}_t^+(x)} = X$.*

3.1. The structure of $S(\mathfrak{I})$. Let $S = S(\mathfrak{I}) \subseteq \Sigma$ be the set given in (3). In general, except in few cases, a definite structure cannot be given for S, though its largeness can be understood in some cases. In fact in the sequel, we give sufficient conditions where S is dense in Σ. First a weaker version of specification property for subshifts:

Definition 3.4. *A subshift Σ is called a* subshift of variable gap length *or SVGL, if there exists $M \in \mathbb{N}$ such that for u and v in $\mathcal{L}(\Sigma)$, there is w with $|w| \leq M$ and $uwv \in \mathcal{L}(\Sigma)$.*

When Σ is mixing and SVGL, then Σ has *specification property* and in this situation, there exists $M \in \mathbb{N}$ such that for u and v in $\mathcal{L}(\Sigma)$ there is w with $|w| = M$ and $uwv \in \mathcal{L}(\Sigma)$. Clearly an SVGL is irreducible. Moreover, all sofics are SVGL; however, there are SVGL's which are not sofic [10].

Proposition 3.5. *Let $\mathfrak{I} = (X, \mathcal{F}, \Sigma)$ be transitive along some $\sigma \in \Sigma$, $\mathcal{F}$ surjective and Σ an SVGL. Then, S defined in (3) is dense in Σ. If $S \neq \Sigma$, then $\Sigma \setminus S$ is also dense in Σ.*

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