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On Einstein Lorentzian Lie groups: type (a1)

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Abstract. The harmonicity vector fields on Einstein Lorentzian Lie groups type (a1) are determined. Left-invariant vector fields defining harmonic maps are also classified.

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1. Introduction

In [1] it is proved that a (simply-connected) four-dimensional homogeneous Riemannian manifold is either symmetric or isometric to a Lie group equipped with a left-invariant Riemannian metric. Indeed, the class of n-dimensional simply connected Lorentzian Lie groups (respectively ,Lorentzian Lie algebras) coincides with the class of the Riemannian ones. Using this fact, four-dimensional Einstein Lorentzian Lie groups have been classified [3]. On the other hand, investigating critical points of the energy associated to vector fields is an interesting problem from different points of view. In Riemannian settings, it has been proved that critical points of the energy functional $E : \mathfrak{X}(M) \to R$, restricted to maps defined by vector fields, are parallel vector fields [7–10]. Moreover, Gil-Medrano [7] studied when V is a harmonic map. So, it is natural to determine the harmonicity properties of vector fields on four-dimensional Lorentzian Einstein Lie groups.

A Riemannian manifold admitting a parallel vector field is locally reducible, and the same is true for a pseudo-Riemannian manifold admitting an either space-like or time-like parallel vector field. This leads us to consider different situations, where some interesting types of non-parallel vector fields can be characterized in terms of harmonicity properties [2, 6, 8, 9].

If $V : (M, g) \longrightarrow (TM, g^s)$ is a critical point for the energy functional, then V is said to define a harmonic map. The Euler-Lagrange equations characterize vector fields V defining harmonic maps as the ones whose tension field $\theta(V) = tr(\nabla^2 V)$ vanishes. Consequently, V defines a harmonic map from (M, g) to (TM, g^s) if and only if

$$tr[R(\nabla_{.}V, V).] = 0, \quad \nabla^*\nabla V = 0, \tag{1}$$

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where with respect to a pseudo-orthonormal local frame $\{e_1, ..., e_n\}$ on (M, g), with $\varepsilon_i = g(e_i, e_i) = \pm 1$ for all indices i, one has

$$\nabla^*\nabla V = \sum_i \varepsilon_i (\nabla_{e_i}\nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V).$$

A smooth vector field V is said to be a harmonic section if it is a critical point of $E^v(V) = (1/2)\int_M ||\nabla V||^2 dv$, where E^v is the vertical energy. The corresponding Euler-Lagrange equations are given by

$$\nabla^*\nabla V = 0. \tag{2}$$

Let $\mathfrak{X}^\rho(M) = \{V \in \mathfrak{X}(M) : ||V||^2 = \rho^2\}$ and $\rho \neq 0$. Then, one can consider vector fields $V \in \mathfrak{X}^\rho(M)$ which are critical points for the energy functional $E|_{\mathfrak{X}^\rho(M)}$, restricted to vector fields of the same constant length. The Euler-Lagrange equations of this variational condition are given by

$$\nabla^*\nabla V \quad \textit{is} \quad \textit{collinear} \quad \textit{to} \quad V. \tag{3}$$

In the non-compact case, the condition (3) is taken as a definition of critical points for the energy functional under the assumption $\rho \neq 0$, that is, if V is not light-like. If $\rho = 0$, then (3) is still a sufficient condition so that V is a critical point for the energy functional $E|_{\mathfrak{X}^0(M)}$, restricted to light-like vector fields ([2], Theorem 26).
Following [3], four-dimensional Einstein Lorentzian Lie groups are classified into four types, denoted by (a1), (a2), (c1) and (c2). In the present paper using a case-by-case argument we shall completely investigat the harmonicity of vector fields on these spaces.

2. Harmonicity of vector fields

Let (G, g) be a four-dimensional Lorentzian Lie group. Following [3], the Lie algebra $\mathfrak{g}$ of G is a semi-direct product $\ltimes \mathfrak{g}_3$, where $= span\{e_4\}$ acts on $\mathfrak{g}_3 = span\{e_1, e_2, e_3\}$, and the Lorentzian inner product on $\mathfrak{g}$ is described by

$$(a) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (c) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In 2013 Calvaruso and Zaeim [3] obtained the following result:

THEOREM 2.1. *Let G be a four-dimensional simply connected Lie group. If g is a left-invariant Lorentzian Einstein metric on G, then the Lie algebra $\mathfrak{g}$ of G is isometric to $\mathfrak{g} = \ltimes \mathfrak{g}_3$, where $\mathfrak{g}_3 = span\{e_1, e_2, e_3\}$ and $= span\{e_4\}$, and one of the following cases occurs.*

(a) $\{e_i\}_{i=1}^4$ *is a pseudo-orthonormal basis, with e_3 time-like. In this case, G is isometric to one of the following semi-direct products $\mathbb{R} \ltimes G_3$:*

(a1) $\mathbb{R} \ltimes H$, *where H is the Heisenberg group and $\mathfrak{g}$ is described by one of the following sets of conditions:*

(1) $[e_1, e_2] = \epsilon A e_1, [e_1, e_3] = A e_1, [e_1, e_4] = \delta A e_1, [e_3, e_4] = -2A\delta(\epsilon e_2 - e_3)$,

(2) $[e_1, e_2] = \frac{\epsilon\sqrt{A^2-B^2}}{2} e_1, [e_1, e_3] = -\frac{\epsilon\delta\sqrt{A^2-B^2}}{2} e_1, [e_1, e_4] = \frac{\delta A+B}{2} e_1, [e_2, e_4] = B(e_2 + \delta e_3), [e_3, e_4] = A(e_2 + \delta e_3)$,

(3) $[e_1, e_2] = \frac{\epsilon A\sqrt{A^2-B^2}}{B} e_1, [e_1, e_3] = \epsilon\sqrt{A^2 - B^2} e_1, [e_2, e_4] = B e_2 - A e_3, [e_3, e_4] = A e_2 - \frac{A^2}{B} e_3$,

(4) $[e_1, e_2] = \epsilon\sqrt{A^2 - B^2} e_1 + B e_2, [e_3, e_4] = A e_3$,

In all the cases listed above, $\epsilon = \pm 1$, $\delta = \pm 1$ and A, B, C, D are real constants.

All four-dimensional simply connected Einstein Lorentzian Lie groups of type (a1) are symmetric [3] and the study of harmonic invariant vector fields on these spaces would be natural and interesting. The main purpose of this section is to investigat the harmonicity properties of left-invariant vector fields on four-dimensional Lorentzian Lie group of type (a1). The following notation is necessary.

Remark 2.2. Let $\tilde{\mathfrak{X}}^\rho(M)$ denote the set of all vector fields $V \in \mathfrak{X}^\rho(M)$, which are critical points for the energy functional $E|_{\mathfrak{X}^\rho(M)}$, restricted to vector fields of the same constant length. Remember that ρ is not necessarily the same for different cases.

Let (G, g) be a four-dimensional Lorentzian Lie group of type (a1) and $\{e_i\}_{i=1}^4$ a pseudo-orthonormal basis, with e_3 time-like. Under these assumptions, we prove the following result.

Theorem 2.3. *Let $\mathfrak{g}$ be the Lie algebra of G and $V = ae_1 + be_2 + ce_3 + de_4 \in \mathfrak{g}$ a left-invariant vector field on G for some real constants a, b, c, d. For the different cases $(1) - (4)$ of type* (a1)*, we have:*

(1) *: $V \in \tilde{\mathfrak{X}}^\rho(G)$ if and only if $V = c(e_2 - e_3 - e_4)$, that is, $b = -c = -d$. In this case $\epsilon = 1$, $\nabla^*\nabla V = 3A^2V$.*
(2) *: $V \in \tilde{\mathfrak{X}}^\rho(G)$ if and only if $V = c(e_2 + e_3 - e_4)$, that is, $b = c = -d$. In this case $\epsilon = -1$, $\delta = 1$, $\nabla^*\nabla V = -\frac{3}{4}(A + B)^2 V$.*
(3) *: $V \in \tilde{\mathfrak{X}}^\rho(G)$, in this case, $\nabla^*\nabla V = -\frac{(A^2-B^2)^2}{B^2}V$.*
(4) *: $V \in \tilde{\mathfrak{X}}^\rho(G)$ if and only if $a = b = 0$, in this case $\nabla^*\nabla V = -A^2V$ or $c = d = 0$, in this case $\nabla^*\nabla V = (B^2 - A^2)V$.*

Proof. The above statement is obtained from a case-by-case argument. As an example, we report the details for case (3) here. Let $V \in \mathfrak{g}$ be a critical point for the energy functional. The components of the Levi-Civita connection are the following:

$$\begin{array}{c} \nabla_{e_1}e_1 = -\frac{\epsilon A\sqrt{A^2-B^2}}{B}e_2 + \epsilon\sqrt{A^2 - B^2}e_3, \quad \nabla_{e_1}e_2 = \frac{\epsilon A\sqrt{A^2-B^2}}{B}e_1 \\ \nabla_{e_1}e_3 = \epsilon\sqrt{A^2 - B^2}e_1 \quad \nabla_{e_2}e_2 = -Be_4, \quad \nabla_{e_2}e_3 = -Ae_4, \\ \nabla_{e_2}e_4 = Be_2 - Ae_3, \qquad \nabla_{e_3}e_2 = -Ae_4, \\ \nabla_{e_3}e_3 = -\frac{A^2}{B}e_4, \qquad \nabla_{e_3}e_4 = Ae_2 - \frac{A^2}{B}e_3, \end{array} \tag{4}$$

while $\nabla_{e_i}e_j = 0$ in the remaining cases. From (4) we obtain

$$\begin{array}{c} \nabla_{e_1}V = \epsilon\sqrt{A^2 - B^2}(\frac{cB+bA}{b}e_1 - \frac{aA}{B}e_2 + ae_3), \\ \nabla_{e_2}V = dBe_2 - dAe_3 - (cA + bB)e_4, \qquad \nabla_{e_4}V = 0 \\ \nabla_{e_3}V = dAe_2 - \frac{dA^2}{B}e_3 - \frac{A(cA+bB)}{B}e_4. \end{array} \tag{5}$$

Clearly, there are no parallel vector fields $V \neq 0$ in $\mathfrak{g}$. We can now calculate $\nabla_{e_i}\nabla_{e_i}V$ and $\nabla_{\nabla_{e_i}e_i}V$ for all indices i and we find

$$\begin{array}{c} \nabla_{e_1}\nabla_{e_1}V = \frac{-(A^2-B^2)}{B^2}(a(A^2 - B^2)e_1 + (cB + bA)(Ae_2 - Be_3)), \\ \nabla_{e_2}\nabla_{e_2}V = -(cB + bA)(Be_2 - Ae_3) + d(A^2 - B^2)e_4, \\ \nabla_{e_3}\nabla_{e_3}V = \frac{-A^2}{B^2}((cB + bA)(Be_2 - Ae_3) + d(A^2 - B^2)e_4), \quad \nabla_{e_4}\nabla_{e_4}V = 0, \\ \nabla_{\nabla_{e_1}e_1}V = \nabla_{\nabla_{e_3}e_3}V = \nabla_{\nabla_{e_2}e_2}V = \nabla_{\nabla_{e_4}e_4}V = 0. \end{array} \tag{6}$$

Thus, we get

$$\begin{aligned} \nabla^*\nabla V = & \sum_i \varepsilon_i(\nabla_{e_i}\nabla_{e_i}V - \nabla_{\nabla_{e_i}e_i}V) = \frac{-(A^2-B^2)}{B^2}(a(A^2 - B^2)e_1 + \\ & (cB + bA)(Ae_2 - Be_3)) - (cB + bA)(Be_2 - Ae_3) + d(A^2 - B^2)e_4 - \\ & (\frac{-A^2}{B^2}((cB + bA)(Be_2 - Ae_3) + d(A^2 - B^2)e_4)) = -\frac{(A^2-B^2)^2}{B^2}V. \end{aligned}$$

TABLE 1. Equivalent properties for the cases $(1)-(4)$ in Theorem 2.3.

(G,g)	Equivalent properties (denoted by $\equiv$)
(1)	V is geodesic; $\equiv$ $V \in \tilde{\mathfrak{X}}^{\rho}(G)$; $\equiv$ none of these vector fields is harmonic (in particular, defines a harmonic map); $\equiv$ $V = c(e_2 - e_3 - e_4)$,
(2)	V is geodesic; $\equiv$ V is harmonic if and only if $A=-B$; $\equiv$ $V \in \tilde{\mathfrak{X}}^{\rho}(G)$; $\equiv$ V defines harmonic map if and only if $A=-B$; $\equiv$ V is Killing if and only if $A=-B$ and $d=0$; $\equiv$ $V = c(e_2 + e_3 - e_4)$,
(3)	V is geodesic if and only if $A=\pm B$ and $b=\mp c$; $\equiv$ V is harmonic if and only if $A=\pm B$; $\equiv$ $V \in \tilde{\mathfrak{X}}^{\rho}(G)$; $\equiv$ V defines harmonic map if and only if $A=\pm B$; $\equiv$ V is Killing if and only if $A=\pm B$, $b=\mp c$ and $d=0$,
(4)	V is geodesic if and only if $a=b=c=0$; $\equiv$ $V \in \tilde{\mathfrak{X}}^{\rho}(G)$ if and only if $a=b=0$; $\equiv$ none of these vector fields is harmonic (in particular, defines a harmonic map).

□

As the definitions already show, V is harmonic if $\nabla^*\nabla V = 0$ and V defines a harmonic map if and only if

$$tr[R(\nabla_{.}V,V).] = 0, \quad \nabla^*\nabla V = 0.$$

For case (3) in Theorem 2.3, $\nabla^*\nabla V = -\frac{(A^2-B^2)^2}{B^2}V = 0$ if and only if $A=\pm B$, that is, V is harmonic if and only if $A=\pm B$. Let R denote the curvature tensor of (M,g), taken with the sign convention $R(X,Y) = \nabla[X,Y] - [\nabla X, \nabla Y]$. Then, using (5), we find

$$\begin{gathered} R(\nabla_{e_1}V,V)e_1 = \tfrac{\epsilon(A^2-B^2)^{3/2}}{B^3}((A^2-B^2)a^2 + (bA+cB)^2)(Ae_2 - Be_3), \\ \tfrac{A^2}{B^2}R(\nabla_{e_2}V,V)e_2 = R(\nabla_{e_3}V,V)e_3 = \tfrac{A^2(A^2-B^2)}{B^3}((A^2-B^2)d^2 - (cA+bB)^2)e_4, \\ R(\nabla_{e_4}V,V)e_4 = 0 \end{gathered}$$

and so, when $A=\pm B$ clearly,

$$tr[R(\nabla_{.}V,V).] = \textstyle\sum_i \varepsilon_i R(\nabla_{e_i}V,V)e_i = 0.$$

Hence, $tr[R(\nabla_{.}V,V).] = 0$ if and only if $A=\pm B$. Appling this argument for other cases of type (a1) proves the following classification result.

THEOREM 2.4. *Let V be a critical point for the energy functional, described by the conditions* (2) *and* (3) *in Theorem 2.3. Then, for cases* (2) *and* (3), *V defines a harmonic map if and only if $A=-B$ and $A=\pm B$ respectively.*

A vector field V is geodesic if $\nabla_V V = 0$, and is Killing if $\mathcal{L}_V g = 0$, where $\mathcal{L}$ denotes the Lie derivative. Parallel vector fields are both geodesic and Killing, and vector fields with these special geometric features often have particular harmonicity properties. A straightforward calculation proves the following main classification result.

COROLLARY 2.5. *If g is a left-invariant Lorentzian Einstein metric on G, then for the cases $(1)-(4)$ in Theorem 2.3, the equivalent properties for $V = ae_1 + be_2 + ce_3 + de_4 \in \mathfrak{g}$ are listed in Table 1.*

REMARK 2.6. Recall that for a Lorentzian Lie group, a left-invariant vector field V is spatially harmonic if and only if

$$(7) \quad \tilde{X}_V = -\nabla^*\nabla V - \nabla_V\nabla_V V - divV \cdot \nabla_V V + (\nabla V)^t \nabla_V V \quad \textit{is collinear to } V.$$

Clearly, conditions (3) and (7) coincide for geodesic vector fields. Hence, the results listed in Table 1 show that for cases (1) and (2), V is spatially harmonic and for case (3), V is spatially harmonic if and only if $A = \pm B$ and $b = \mp c$. For case (4), V is spatially harmonic if and only if $a = b = c = 0$.

References

[1] L. Bérard-Bérgery, Homogeneous Riemannian spaces of dimension four, *S*eminar A.Besse, Four-dimensional Riemannian geometry, (1985).

[2] G. Calvaruso, Harmonicity properties of invariant vector fields on three-dimensional Lorentzian Lie groups, J. Geom. Phys. 61 (2011), 498-515.

[3] G. Calvaruso and A. Zaeim, Four-dimensional Lorentzian Lie groups, Differ. Geom. Appl. 31 (2013), 496-509.

[4] M. Chaichi, E. Garcia-Rio, Y. Matsushita, Curvature properties of four-dimensional Walker metrics, Classical Quantum Gravity, 22 (3), (2005), 559-577.

[5] M. Chaichi, E. Garcia-Rio, Y. Matsushita, Three-dimensional Lorentz manifolds admitting a parallel null vector field, J. Phys. A, 38 (4), (2005), 841-850.

[6] S. Dragomir and D. Perrone, Harmonic Vector Fields: Variational Principles and Differential Geometry, Elsevier, Science Ltd, (2011).

[7] O. Gil-Medrano, Relationship between volume and energy of vector fields, Diff. Geom. Appl. 15 (2001), 137–152.

[8] T. Ishihara, Harmonic sections of tangent bundles, J. Math. Tokushima Univ. 13, (1979), 23–27.

[9] O. Nouhaud, Applications harmoniques d'une variété riemannienne dans son fibré tangent, Généralisation, C. R. Acad. Sci. Paris Sér. A-B, 284 (14), (1977), A815–A818.

[10] A. Zaeim, M. Chaichi, Y. Aryanejad, On Lorentzian Two-Symmetric Manifolds of Dimension-Four, Iranian Journal of Mathematical Sciences and Informatics, 12 (4), (2017) 81-94.