سومین کنفرانس سیستم های دینامیکی و نظریه های هندسی
The 3rd Conference on Dynamical Systems & Geometric Theories
Hakim Sabzevari University
Faculty of Mathematics and Computer Science
Iran, Sabzevar
27 & 28 January 2022
دانشگاه حکیم سبزواری
دانشکده ریاضی و علوم کامپیوتر
۷ و ۸ بهمن ۱۴۰۰

An upper bound for the measure-theoretic pressure of endomorphisms

Maryam Razi[1,*], Pouya Mehdipour [2]Sanaz Lamei[3]

[1]University of Guilan-Iran. .razi264@gmail.com
[2]Federal University of Viçosa-MG, Brazil. Pouya@ufv.br
[3]University of Guilan-Iran. lamei@guilan.ac.ir

Abstract. A measure-theoretic pressure was defined by [L. He, J. Lv and L. Zhou, Definition of measure-theoretic pressure using spanning sets, *Acta Math. Sinica*(English Series), **20** (2004), 709-718] based on Katok entropy formula. In this article, we investigate an upper bound for the measure theoretic pressure of a C^2 endomorphism on a closed s-dimensional Riemannian manifold (compact and boundaryless) preserving a hyperbolic Borel probability measure.

Keywords: Thermodynamic formalism, Periodic points, Measure Pressure, Non-uniform hyperbolicity.

AMS Mathematics Subject Classification [2020]: 18A32, 18F20, 05C65 (at least 1 and at most 3)

---

1. Introduction

Topological and measure theoretical pressures, as generalizations of the topological and measure theoretical entropy, are significant quantities in Ergodic Theory and Statistical Mechanics. They provide a determinant to measure the local complexity of dynamics defined in compact spaces. In this paper we deal with measure theoretical pressure for endomorphisms which are C^2 local diffeomorphism cascades on closed s-dimensional Riemannian manifolds (compact and boundaryless). It is considered the endomorphisms with fixed index, i.e., the dimension of the local unstable manifolds is fixed. Note that when studying endomorphisms, different local dynamics are involved and this can cause a variety of unstable indices at different points. Our aim is to extend the results of [2] which uses the Katok entropy formula [4] to establish an upper bound for measure theoretic pressure of diffeomorphisms in terms of their periodic points. We try to formulate such upper bound for endomorphisms. The difficulty in achieving this result for the case of endomorphism was due to the lack of a version of Katok Closing Lemma, for endomorphisms. In [5] a version of Closing Lemma under the priory mentioned conditions for endomorphisms is

*Speaker. Email address: name@email.com, obtained. These results together with Theorem 2.1 of [2] on measure-theoretic pressures, defined on spanning sets [1], are the crucial implements used for the proof of the following Theorem.

1.1. Multiplicative Ergodic Theorem for Natural Extension. We devote this subsection to introduce the relevant Multiplicative Ergodic Theorem for Natural Extension ($\widetilde{MET}$) and the Katok Closing Lemma for C^2 local diffeomorphisms. These results play a crucial role for our purpose.

THEOREM 1.1 ([6]). *Let μ be an $f-$invariant Borel probability measure on M. We denote by $\tilde{\mu}$ the $\tilde{f}-$invariant Borel probability measure on M^f such that $\pi_*\tilde{\mu} = \mu$. There exists a full measure subset $\tilde{\mathcal{R}}$ called set of regular points such that for all $\tilde{x} = (x_n) \in \tilde{\mathcal{R}}$ and $n \in \mathbb{Z}$ the tangent space $T_{x_n}M$ splits into a direct sum*

$$T_{x_n}M = E_1(\tilde{x}, n) \oplus \cdots \oplus E_{r(x_0)}(\tilde{x}, n)$$

and there exists $-\infty < \lambda_1(\tilde{x}) < \cdots < \lambda_{r(\tilde{x})} < \infty$ and $m_i(\tilde{x})$ $(i = 0, 1, ..., r(\tilde{x}))$ such that:

(1) $\dim E_i(\tilde{x}, n) = m_i(\tilde{x})$;
(2) $D_{x_n}f(E_i(\tilde{x}, n)) = E_i(\tilde{x}, n+1)$, *and* $D_{x_n}f|_{E_i(\tilde{x},n)} : E_i(\tilde{x}, n) \to E_i(\tilde{x}, n+1)$ *is an isomorphism. For* $v \in E_i(\tilde{x}, n)\backslash\{0\}$,
$$\begin{cases} \lim\limits_{m\to\infty} \frac{1}{m} \log \|D_{x_n}f^m(v)\| = \lambda_i(\tilde{x}); \\ \lim\limits_{m\to\infty} -\frac{1}{m} \log \|(D_{x_{n-m}}f^m|_{E_i(\tilde{x},n-m)})^{-1}(v)\| = \lambda_i(\tilde{x}); \end{cases}$$
(3) *if $i \neq j$ then*
$$\lim_{n\to\pm\infty} \frac{1}{n} \log \sin \angle(E_i(\tilde{x}, n), E_j(\tilde{x}, n)) = 0,$$
where $\angle(V, W)$ denotes the angle between sub-spaces V and W of $T_{x_n}M$.
(4) *$r(.)$, $\lambda_i(.)$ and $m_i(.)$ are measurable and $\tilde{f}-$invariant. Moreover $r(\tilde{x}) = r(x_0)$, $\lambda_i(\tilde{x}) = \lambda_i(x_0)$ and $m_i(\tilde{x}) = m_i(x_0)$ for all $i = 1, 2, ..., r(\tilde{x})$.*

Let μ be an $f-$invariant Borel probability measure on M. By $\widetilde{MET}$, there exists a full measure subset $\tilde{\mathcal{R}}$ called *"Lyapunov regular set"*. The assumption that the measure mu is ergodic and hyperbolic imposes that Lyapunov exponents are constant almost everywhere. The set of Lyapunov regular points without zero Lyapunov exponents, contains a non-uniformly hyperbolic set of full $\tilde{\mu}-$measure with

$$\lambda = \lambda^\mu, \;\; \theta = \theta^\mu, \;\; C(\tilde{x}) = C(\tilde{x}, \epsilon), \;\; K(\tilde{x}) = K(\tilde{x}, \epsilon)$$

where $\lambda = \lambda^\mu$ (resp $\theta = \theta^\mu$) is the least in modulus positive (resp. negative) Lyapunov exponent. Suppose that μ has k positive Lyapunov exponents, then the index of f is considered k. Let (f, μ) be a measure dynamics with μ a non-atomic hyperbolic ergodic measure. Without loss of generality from now on, we set $\tilde{\mathcal{R}}$ as the non-uniformly hyperbolic subset of the Lyapunov regular points, with full $\mu-$measure. We denote its projection on M by $\mathcal{R} = \pi(\tilde{\mathcal{R}})$.

DEFINITION 1.2 (**Pesin Blocks**). Fix $0 < \epsilon \ll 1$. For any $l > 1$, we define a Pesin block $\tilde{\Delta}_l$ of M^f consisting of $\tilde{x} = (x_n) \in M^f$ for which there exists a sequence of splittings $T_{x_n}M = E^s(\tilde{x}, n) \oplus E^u(\tilde{x}, n)$, $n \in \mathbb{Z}$, satisfying:

- $\dim E^s(\tilde{x}, n) = k$;
- $D_{x_n}f(E^s(\tilde{x}, n)) = E^s(\tilde{x}, n+1)$, $D_{x_n}f(E^u(\tilde{x}, n)) = E^u(\tilde{x}, n+1)$;

- for $m \geq 0$, $v \in E^s(\tilde{x}, n)$ and $w \in E^u(\tilde{x}, n)$;

$$\begin{cases} \|D_{x_n} f^m(v)\| \leq e^l e^{-(\theta-\epsilon)m} e^{(\epsilon|n|)} \|v\|, \forall n \in \mathbb{Z}, n \geq 1 \\ \|(D_{x_{n-m}} f^m|_{E^u(\tilde{x}, n-m)})^{-1}(w)\| \leq e^l e^{-(\lambda-\epsilon)m} e^{(\epsilon|n-m|)} \|w\|, \forall n \in \mathbb{Z}, n \geq 1; \end{cases}$$

- $\sin \angle(E^s(\tilde{x}, n),\, E^u(\tilde{x}, n)) \geq e^{-l} e^{-\epsilon|n|}$.

The notation $\tilde{\Delta}_l^k$ represents a Pesin Block with index k being the dimension of the local unstable manifold.

Note that, Pesin blocks are compact subsets of M^f. The sub-spaces $E^s(\tilde{x}, n)$ and $E^u(\tilde{x}, n)$ of $T_{x_n}M$ depend continuously on $\tilde{x}$ and $\tilde{f}^{\pm}(\tilde{\Delta}_l) \subset \tilde{\Delta}_{l+1}$.

In a non-uniformly hyperbolic setting, for C^2 endomorphisms with fixed index, we have the following version of Katok Closing Lemma.

Lemma 1.3 (**Katok Closing Lemma** [5]). *Let f be a C^2-endomorphism of a compact Riemannian $s-$dimensional manifold M. Then, for positive numbers χ, l, δ and $0 < k \leq s$, there exists a number $\varrho = \varrho(\chi, l, \delta, k) > 0$ such that, if for some point $\tilde{x} \in \tilde{\Delta}_l^k$ and some integer m, one has*

$$\tilde{f}^m(\tilde{x}) \in \tilde{\Delta}_l^k \quad \text{and} \quad \tilde{d}(\tilde{x}, \tilde{f}^m(\tilde{x})) < \varrho, \tag{1.1}$$

then, there exists a point $z \in M$ and $\bar{z} \in M^f$, such that $z = \pi(\bar{z})$), and

- *$f^m(z) = z$ and $\tilde{f}^m(\bar{z}) = \bar{z}$;*
- *$\tilde{d}(\tilde{x}, \bar{z}) < \delta$ and so $d(x, z) < \delta$;*
- *the point z is a hyperbolic periodic point for f and its $W^s_{loc}(x)$ and $W^u_{loc}(\bar{z})$ manifolds are admissible manifolds near the point x,.*

2. Meassure theoretic pressure

Let M be a compact n-dimensional Riemannian manifold and $f : M \to M$ be a C^2 local diffeomorphism preserving an ergodic hyperbolic measure μ (non-atomic). For any $\tilde{x} \in M^f$ with $\pi(\tilde{x}) = x$, let $r(x)$ denote the number of Lyapunov exponents of f at point $x \in M$, $\lambda_i(x)$ $(1 \leq i \leq r(x) \leq s)$ the $i-$th Lyapunov exponent and $k_i(x)$ its multiplicity. Due to Theorem 1.1, and the ergodicity of the measure, the $r^\mu, \lambda_i^\mu, k_i^\mu$ are constant.

Let $f : M \to M$ be a C^2 map and μ an f-invariant Bore1 probability measure on M with $\tilde{\mu}$ the corresponded measure on M^f. Then, the following is a metric on M. This metric is called the d_n metric.

$$d_n(x, y) = \max_i \{d(f^i(x), f^i(y));\ 0 < i \leq n,\ x, y \in M\}.$$

For $\phi \in C(M)$ and $\mu \in \mathcal{M}_f(M)$ the measure theoretic pressure with respect to the measure entropy $h_\mu(f)$ for a definition) is defined as following.

$$P_\mu(f, \phi) = h_\mu(f) + \int \phi \, d\mu. \tag{2.1}$$

In [2], the authors provide a definition for measure theoretical pressure using measurable sets. The definition is stated as follows. Denote by $B_n(x, \epsilon)$ the ϵ-ball centered on x in d_n-metric. For $\epsilon > 0$ a set $E \subset M$ is said to be an (n, ϵ)-spanning set, if $M \subset \bigcup_{x \in E} B_n(x, \epsilon)$. For $\rho > 0$ one can define the $\mu-$ (n, ϵ, ρ)-spanning set if $\mu(\bigcup_{x \in E} B_n(x, \epsilon)) > 1 - \rho$.

A set $F \subset M$ is said to be an (n, ϵ)-separated set, if for $x \neq y \in F$ there exists some $0 \leq i < n$ such that $d(f^i(x), f^i(y)) \geq \epsilon$. For $\rho > 0$, the $\mu-$ (n, ϵ, ρ)-separated sets are defined similarly. Note that by definition, any (n, ϵ)-separating set is an (n, ϵ)-spanning set.

In what follows the notations $\underline{\lim}$ and $\overline{\lim}$ are used to represent respectively $\liminf_{n\to\infty}$ and $\limsup_{n\to\infty}$. Let $C(M)$ be the space of all continuous real valued functions on M. For $\phi \in C(M)$, one define $S_n\phi(x) = \sum_{i=0}^{n-1} \phi(f^i(x))$ and

$$P(f, \phi, n, \epsilon) = \inf\{\sum_{x\in E} \exp\ S_n(\phi)|\, E\ \ is\ (n, \epsilon)\text{-}\ spanning\ set\},$$

and for $\rho > 0$;

$$P^*(f, \phi, n, \epsilon, \rho) = \inf\{\sum_{x\in E} \exp\ S_n(\phi)|\, E\ \ is\ \mu\text{-}(n, \epsilon, \rho)\text{-}spanning\ set\}.$$

Then, the **Topological Pressure** of f is defined as the map $P(f, .) : C(M) \to \mathbb{R}$, where

$$P(f, \phi) = \lim_{\epsilon\to 0} \overline{\lim}_{n\to\infty} \frac{1}{n} \log P(f, \phi, n, \epsilon).$$

In a similar way the **Measure Theoretic Pressure** of f with respect to μ is defined as,

$$P^*_\mu(f, \phi) = \lim_{\rho\to 0} \lim_{\epsilon\to 0} \overline{\lim}_{n\to\infty} \frac{1}{n} \log P(f, \phi, n, \epsilon, \rho). \tag{2.2}$$

The following Theorem, establishes the equality of (2.2 and 2.1).

THEOREM 2.1 (Theorem 2.1 of [2]). *Suppose (X, d) is a compact metric space and $f : X \to X$ be a continuous map. For any $\mu \in \mathcal{E}(f)$ and $\varphi \in C(X)$,*

$$Q^*_\mu(f, \varphi) = P^*_\mu(f, \varphi) = P_\mu(f, \varphi) = h_\mu(f) + \int \varphi d\mu,$$

where $Q^*_\mu(f, \phi) = \lim_{\epsilon\to 0} \underline{\lim}_{n\to\infty} \frac{1}{n} \log P(f, \phi, n, \epsilon)$.

The following Variational Principle assigns the relation between the topological and measure theoretical pressure.

THEOREM 2.2. *[4] Let $f : X \to X$ be a continuous map and $\varphi \in C(X)$. Then*

$$P(f, \varphi) = \sup\{P_\mu(f, \varphi) \mid \mu \in \mathcal{M}(f)\} = \sup\{P_\mu(f, \varphi) \mid \mu \in \mathcal{E}(f)\}.$$

In this work the following upper bound for the measure theoretic pressure of a C^2 endomorphism is investigated.

THEOREM 2.3. *Let $f : M \to M$ be a C^2 local diffeomorphism on a compact s-dimensional Riemannian manifold M (with fixed index), and μ a hyperbolic measure. Then for any $\phi \in C(M)$,*

$$P_\mu(f, \phi) \leq \lim_{n\to\infty} \sup \frac{1}{n} \log \sum_{x\in Fix(f^n)} \exp(s_n(\phi)(x)).$$

3. Conclusion

In this paper we give an upper bound for the measure theoretic pressure of a C^2 endomorphism on a closed s-dimensional Riemannian manifold (compact and boundaryless) preserving a hyperbolic Borel probability measure with fixed index.

References

1. R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Springer Lecture Notes in Mathematics 470. Springer, Berlin, 1975.
2. L. He, J. Feng LV, L. Zhou, Definition of Measure-theoretic Pressure Using Spanning Sets, *Acta Mathematica Sinica,* Vol.20, No.4, pp. 709-718, 2004.
3. A. Katok, Lyapunov exponents, entropy and periodic orbits for diffeomorphisms. *Inst. Hautes Études Sci. Publ. Math.*, No. 51, 137-173, 1980.
4. Anatole Katok; Boris Hasselblatt; *Introduction to the modern theory of dynamical systems.* (English summary) With a supplementary chapter by Katok and Leonardo Mendoza. Encyclopedia of Mathematics and its Applications, 54. Cambridge University Press, Cambridge, 1995. New spaces in physics, Cambridge Univ. Press, Cambridge, 7 (2021), 327–372.
5. P. Mehdipour, Katok closing lemma for non-singular endomorphisms, *Results in Mathematics,* volume 75, Article number: 66, 2020.
6. Min, Qian, Jian-sheng, Xie; Shu, Zhu *Smooth ergodic theory for endomorphisms.* Lecture notes in mathematics, Vol. 1978, Springer-Verlag, Berlin Heidelberg, 2009.