Synchronized systems and entropy minimality

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Abstract. Let X be an entropy minimal synchronized system and $\varphi : X \to Y$ a factor code. We show that Y is synchronized whenever φ is entropy preserving. With this property, entropy preserving is equivalent to having a degree. Moreover, entropy minimality is equivalent to X being intrinsically ergodic of full support and in this situation, the entropy of X is identical with the synchronized entropy of X.

Keywords: shift of finite type, sofic, coded, entropy, degree

AMS Mathematics Subject Classification [2020]: 37B10, 37B40, 37A05

1. Introduction

Entropy minimality was introduced by Coven and Smtal [2] as a property of dynamical systems which is stronger than topological transitivity and weaker than minimality. There has been some recent work which describes some conditions which are equivalent to entropy minimality for shifts of finite type. Our goal of this note is to look for entropy minimality among the synchronized systems which are a well-known subclass of coded systems.

As any other topological dynamical system, the study of possible measures preserved by the shift map is of interest in coded systems. In particular, the investigation for the existence and uniqueness of a measure of maximal entropy has a long history and those systems with this unique invariant measure are called *intrinsically ergodic*. This measure, if exists, is the most natural measure on subshifts and is the main tool for studying their statistical properties.

Parry [5] established intrinsically ergodic for topologically transitive shifts of finite type and all their subshift factors (sofic shifts) and Bowen [1] proved for shifts with specification property. We extend their results and will show that a synchronized system (X, σ) with positive entropy is intrinsically ergodic of full support if and only if it is entropy minimal.

In Theorem 3.2, we will prove that entropy minimality is a property invariant by entropy preserving factor codes. Also, we will show that if X is an entropy minimal

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synchronized system, then any factor of X by an entropy preserving factor code will be synchronized (Theorem 3.3). Furthermore, for entropy minimal synchronized systems, entropy preserving is equivalent to having a degree (Theorem 3.5).

Finally, for a synchronized system X with the underlying graph G for its Fischer cover, we show that $h(G) = h_{\text{syn}}(X)$ and if X is entropy minimal or equivalently if X is intrinsically ergodic of full support, then $h(X) = h_{\text{syn}}(X)$ [Theorem 3.8].

2. Background and Notations

The notations has been borrowed from [4] and a brief reminder of the main definitions of symbolic dynamics has been brought here. Let $\mathcal{A}$ be a finite set of alphabet. A *full $\mathcal{A}$-shift* is defined as $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ where the *shift map* σ is defined by $\sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$. Any closed invariant set X of $\mathcal{A}^{\mathbb{Z}}$ is called a *subshift* or a *shift space*. Let $\mathcal{B}_n(X)$ denote the set of all admissible n words, i.e. words of length n and set $\mathcal{B}(X) := \cup_{n\in\mathbb{N}}\mathcal{B}_n(X)$ to be the language of X.

Recall the definition of an $(m+n+1)$-*block map* from [4, §1.5]; however, without loss of generality we will use only one-block maps ($m = n = 0$) which induces a map φ called *code*. Thus if φ is a code from X into another shift space, then $\varphi(\cdots x_{-1}x_0x_1\cdots) = (\cdots\Phi(x_{-1})\Phi(x_0)\Phi(x_1)\cdots)$ where Φ is the $(m+n+1)$-block map.

A *factor code* is an onto code. A code $\varphi : X \to Y$ is finite-to-one if there is an integer M such that $|\varphi^{-1}(y)| \leq M$ for every $y \in Y$. A point x in a shift space X is doubly transitive if every word in X appears in x infinitely many often to the left and to the right. We denote by $D(X)$ the set of doubly transitive points of X.

A shift space X is *irreducible* if for every pair of words $u, v \in \mathcal{B}(X)$ there is a word $w \in \mathcal{B}(X)$ so that $uwv \in \mathcal{B}(X)$. A word $v \in \mathcal{B}(X)$ is *synchronizing* if whenever uv, $vw \in \mathcal{B}(X)$, then $uvw \in \mathcal{B}(X)$.

Let G be a graph with edge set $\mathcal{E}$. The *edge shift* X_G is the shift space specified by $X_G = \{\xi = (\xi_i)_{i\in\mathbb{Z}} \in \mathcal{E}^{\mathbb{Z}} : t(\xi_i) = i(\xi_{i+1}) \,\text{for all}\, i \in \mathbb{Z}\}$. A *labeled graph* $\mathcal{G}$ is a pair $(G, \mathcal{L})$ where G is a graph with the labeling $\mathcal{L} : \mathcal{E} \to \mathcal{A}$. Let $\mathcal{L}_\infty(\xi)$ be the label of a bi-infinite path $\xi \in X_G$. Set $X_{\mathcal{G}} := \{\mathcal{L}_\infty(\xi) : \xi \in X_G\}$ which will be denoted by $\mathcal{L}_\infty(X_G)$ as well. If there is a graph G such that $X = \overline{X_{\mathcal{G}}}$, then we say $\mathcal{G}$ is a *presentation* (or *cover*) of $X_{\mathcal{G}}$.

A shift space is *sofic* if there is a finite graph G such that $X = X_{\mathcal{G}}$. Equivalently X is sofic, if it is a factor of an SFT, or shift of finite type (shifts characterized by a set of finite forbidden words). A labeled graph $\mathcal{G} = (G, \mathcal{L})$ is *right-resolving* if for any vertex I of G and any symbol $a \in \mathcal{A}$, there is at most one edge labeled by a and going out of I. A *minimal right-resolving cover* (or *Fischer cover*) of a sofic shift X is a right-resolving cover of X having the fewest vertices among all right-resolving covers of X.

The entropy of a subshift X is defined by $h(X) = \lim_{n\to\infty}(1/n)\log|\mathcal{B}_n(X)|$. There are some other entropies which will be used in this note. One is the entropy related to graphs given by Gurevich and that is defined as follows. Let G be a connected oriented graph. Then for any vertices I, J

$$h(G) = \lim_{n\to\infty}\frac{1}{n}\log B_{IJ}(n) \tag{1}$$

where $B_{IJ}(n)$ is the number of paths of length n which starts at I and ends at J.

3. Main results

Definition 3.1. A topological dynamical system (X, T) is said to be entropy minimal if all closed T-invariant subsets of X have entropy strictly less than (X, T).

First, we investigate some dynamical properties of a subshift satisfying entropy minimality. The next two theorems deals with the application of entropy preserving factor codes in such systems. Also see Theorem 3.5.

THEOREM 3.2. *Suppose $\varphi : X \to Y$ is an entropy preserving factor code and assume that X is entropy minimal. Then, Y is entropy minimal as well.*

PROOF. Let Z be a proper subshift of Y and assume that $h(Z) = h(Y)$. By surjectivity, $\varphi^{-1}(Z)$ is a proper subshift of X and since X is entropy minimal, $h(\varphi^{-1}(Z)) < h(X)$. Now Z is a factor of $\varphi^{-1}(Z)$ and so $h(Z) \le h(\varphi^{-1}(Z)) < h(X) = h(Y)$ violating our assumption. □

THEOREM 3.3. *Suppose X is an irreducible entropy minimal shift space and let $\varphi : X \to Y$ be an entropy preserving factor code. Then, $\varphi^{-1}(D(Y)) = D(X)$. In particular, if X is synchronized, then φ has a degree and Y is synchronized as well.*

PROOF. For the first part a similar result holds for irreducible sofic shifts [4, Lemma 9.1.13]. The main ingredients for the proof of that result is to have X compact, φ entropy preserving and the fact that entropy minimality holds for irreducible sofics [4, Corollary 4.4.9]. All of them are provided here.

The second part is a direct application of [3, Theorem 3.3] and [3, Theorem 4.2]. □

THEOREM 3.4. *If X is an entropy minimal synchronized system with Fischer cover $\mathcal{G} = (G, \mathcal{L})$, then $h(X) = h(G)$.*

PROOF. One has $h(X) = \max\{h(G), h(\partial X)\}$ [6, Theorem 6.16]. Also, ∂X is a proper subsystem of X, and so $h(X) = h(G)$. □

THEOREM 3.5. *Let X be an entropy minimal synchronized system and $\varphi : X \to Y$ a factor code. Then, φ is entropy preserving if and only if it has a degree.*

PROOF. By Theorem 3.3 necessity is at hand, so we prove sufficiency. Assume φ has a degree. Since X is entropy minimal, $h(X) = h(G_X)$ where G_X is the underlying graph of Fischer cover of X (Theorem 3.4). So, $h(X) = h(G_X) = h(G_Y) = h(Y)$. □

THEOREM 3.6. *Suppose X is a subshift with positive topological entropy. Then, any invariant measure on X with maximal entropy is of full support if and only if X is entropy minimal.*

PROOF. First let μ_X be the invariant measure on X with maximal entropy of full support; so $h(X) = h_{\mu_X}$. Suppose Y is a proper subsystem of X and $h(Y) = h(X)$. By the variational principle

$$h(Y) = \sup\{h_\nu : \nu \in \mathcal{M}(Y, \sigma)\} \tag{2}$$

where $\mathcal{M}(Y, \sigma)$ is the set of all invariant measures. The shift map σ is expansive and so there is a measure ν_Y with $h(Y) = h_{\nu_Y}$. Set $\nu_X(A) = \nu_Y(A \cap Y)$ for $A \in \mathcal{M}(X)$ and notice that ν_X is an invariant measure on X vanishing at the open set $X \setminus Y$. Now by a direct verification, $h_{\nu_X} = h_{\nu_Y} = h(Y) = h(X)$ which is absurd by the hypothesis.

For the converse assume that X is entropy minimal and let μ be the measure with maximal entropy which is not of full support. Then there has to be a cylinder $[u] \subset X$ with $\mu([u]) = 0$. Now $Y = X \setminus \cup_{i=-\infty}^{\infty} \sigma^{-i}([u])$ is a closed invariant subset of X and in fact a proper subsystem of X with $\mu(Y) = \mu(X)$. Restrict μ to Y and call it μ_Y. Then, $h_{\mu_Y}(Y) = h_\mu(X) = h(X)$ and this in turn by applying (2) implies that $h(Y) = h(X)$ which violates our assumption. □

THEOREM 3.7. *Let X be a subshift with positive topological entropy. If X is intrinsically ergodic of full support, then X is entropy minimal. The converse is true whenever X is synchronized.*

PROOF. Apply Theorem 3.6 and the fact that a maximal measure of full support for synchronized systems is unique [6]. □

Let X be synchronized and fix a synchronizing word $\alpha \in \mathcal{B}(X)$. Let $C_n(\alpha)$ be the set of words $v \in \mathcal{B}_n(X)$ such that $\alpha v \alpha \in \mathcal{B}(X)$. Then the *synchronized entropy* $h_{\text{syn}}(X)$ is defined by

$$h_{\text{syn}}(X) = \limsup_{n\to\infty} \frac{1}{n} \log |C_n(\alpha)|.$$

This value is independent of α and $h(X) \geq h_{\text{syn}}(X)$. In general, $h(X) \neq h_{\text{syn}}(X)$; however, Thomsen showed that for irreducible sofic shifts $h(X) = h_{\text{syn}}(X)$. Later Jung extended this result to SVGL shifts.

THEOREM 3.8. *Let X be a synchronized system with Fischer cover $\mathcal{G} = (G, \mathcal{L})$. Then $h(G) = h_{\text{syn}}(X)$. In particular, when X is entropy minimal or equivalently if X is intrinsically ergodic of full support, then $h(X) = h_{\text{syn}}(X)$.*

PROOF. First we show that $h(G) \leq h_{\text{syn}}(X)$. Let $\alpha \in \mathcal{B}(X)$ be a synchronizing word. Then all elements of $\mathcal{L}^{-1}(\alpha)$ have the same terminal vertex, say I and suppose L_n denote the set of cycles in X_G of length n starting and terminating at I. Set $\pi \in L_n(I)$ and $\mathcal{L}(\pi) = v$. Also, let $\min\{|w| : \alpha w \alpha \in \mathcal{B}(X)\} = k$ and $w' \in C_k(\alpha)$. Then $vw' \in C_{n+k}(\alpha)$. Since $\mathcal{L}_\infty$ is right-resolving,

$$\mathcal{L} : L_n(I) \longrightarrow C_{n+k}(\alpha)$$

is injective. So,

$$\limsup_{n\to\infty} \frac{1}{n} \log L_n(I) \leq \limsup_{n\to\infty} \frac{1}{n} \log C_{n+k}(\alpha) = h_{\text{syn}}(X).$$

By (1), $\limsup_{n\to\infty} \frac{1}{n} \log L_n(I) = h(G)$. So $h(G) \leq h_{\text{syn}}(X)$.

The converse is quite similar! Let $|\alpha| = l$ and $v \in C_n(\alpha)$ with $\pi \in \mathcal{L}^{-1}(v)$. Since all elements of $\mathcal{L}^{-1}(\alpha)$ have the same terminal vertex I, then for some $\pi' \in \mathcal{L}^{-1}(\alpha)$, $\pi\pi'$ is a cycle in G starting and terminating at I. So

$$h_{\text{syn}}(X) = \limsup_{n\to\infty} \frac{1}{n} \log C_n(\alpha) \leq \limsup_{n\to\infty} \frac{1}{n} \log L_{n+l}(I).$$

Now suppose X is entropy minimal. Then by Theorem 3.4, $h(X) = h(G)$ and hence by above result, $h(X) = h_{\text{syn}}(X)$.

□

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