A Stabilized diagonal-preservin of C*-algebras

Arezoo Hosseini[1*],

[1]Faculty of Mathematics, College of Science, Farhangian University, Tehran, Iran

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ABSTRACT. We give a stabilized version of any *-isomorphism $O_X \to O_Y$ which maps $C(X)$ onto $C(Y)$ is in fact diagonal-preserving under mild conditions on X and Y.

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1. Introduction

Let X and Y be one-sided shift spaces. A *-isomorphism $\Psi : O_X \to O_Y$ is *diagonal-preserving* if $\Psi(D_X) = D_Y$. In this paper we prove that a *-isomorphism $\Psi : O_X \to O_Y$ satisfying $\Psi(C(X)) = C(Y)$ is diagonal-preserving is stabilized. First we need some preliminary results. Everyone can read more in [1, 2].

2. Main Section

LEMMA 2.1. *Let X be a one-sided shift space. Then*

$$C^*(Iso(\mathcal{G}_X)^\circ) = D'_X \subseteq C(X)'.$$

If X contains a dense set of aperiodic points, then $D'_X = C(X)'$.

PROOF. Let $\iota \in C_c(\mathcal{G}_X)$. The condition that $\iota \star g = g \star \iota$ for all $g \in D_X$ means that ι is supported on elements $\gamma \in \mathcal{G}_X$ with $s(\gamma) = r(\gamma)$. It follows that $C^*(Iso(\mathcal{G}_X)^\circ) = D'_X$. The inclusion $D'_X \subseteq C(X)'$ follows from the inclusion $C(X) \subseteq D_X$. □

Consider the equivalence relation $\sim$ on the space $\tilde{X} \times \mathbb{T}$ given by $(\tilde{x}, \iota) \sim (\tilde{y}, \theta)$ if and only if $\tilde{x} = \tilde{y}$ and $\iota^p = \theta^p$ for all $p \in Stab(\tilde{x})$. Then the quotient $\tilde{X} \times \mathbb{T}/\sim$ is compact and Hausdorff and as we shall see (homeomorphic to) the spectrum of $C^*(Iso(\mathcal{G}_X)^\circ)$. We read more in groupoid [3].

LEMMA 2.2. *Let $\sim$ be the equivalence relation on $\tilde{X} \times \mathbb{T}$ defined above. There is a *-isomorphism*

$$\Omega : C^*(Iso(\mathcal{G}_X)^\circ) \to C(\tilde{X} \times \mathbb{T}/\sim),$$

*Speaker. Email address: a.hosseini@cfu.ac.ir,

given by

$$\Omega(f)([\tilde{x},\iota]) = \sum_{p\in Stab(\tilde{x})} f(\tilde{x},p,\tilde{x})\iota^n \tag{1}$$

for $f \in C_c(Iso(\mathcal{G}_X)^\circ)$ *and* $[\tilde{x},\iota] \in \tilde{X} \times \mathbb{T}$.

3. Main Section

THEOREM 3.1. *Let X and Y be one-sided shift spaces with dense sets of aperiodic points and let $\Psi : O_X \to O_Y$ be a *-isomorphism satisfying $\Psi(C(X)) = C(Y)$. Then $\Psi(D_X) = D_Y$.*

PROOF. If $\Psi : O_X \to O_Y$ is a *-isomorphism satisfying $\Psi(C(X)) = C(Y)$, then $\Psi(C(X)') = C(Y)'$. By Lemmas 2.1 and 2.2, there is a homeomorphism

$$h : \tilde{X} \times \mathbb{T}/\sim \to \tilde{Y} \times \mathbb{T}/\sim,$$

such that $\Psi(f) = f \circ h^{-1}$ for $f \in C(\tilde{X} \times \mathbb{T}/\sim)$.
Define the map $q_X : \tilde{X} \times \mathbb{T}/\sim \to \tilde{X}$ by $q_X([\tilde{x},z]) = \tilde{x}$. This is well-defined, continuous and surjective. Furthermore, q_X induces the inclusion $D_X \subseteq C(X)'$. Let $\tilde{x} \in \tilde{X}$ and put $\tilde{y}_{\tilde{x}} = q_Y(h([\tilde{x},1])) \in \tilde{Y}$. The connected component of any $[\tilde{x},z]$ is the set $[\tilde{x},w] \mid w \in \mathbb{T}$, so since any homeomorphism will preserve connected components, we have

$$h(q_X^{-1}(\tilde{x})) = q_Y^{-1}(h([\tilde{x},1])).$$

We may now define a map $\tilde{h} : \tilde{X} \to Y$ by

$$\tilde{h}(\tilde{x}) = \tilde{y}_{\tilde{x}} = q_Y(h([\tilde{x},1]))$$

for $\tilde{x} \in \tilde{X}$, which is well-defined, continuous and surjective. The above considerations show that h is also injective. As both $\tilde{X}$ and $\tilde{Y}$ are compact and Hausdorff, $\tilde{h}$ is a homeomorphism. The relation $\tilde{h} \circ q_X = q_Y \circ h$ ensures that that $\Psi(D_X) = D_Y$ as wanted. □

COROLLARY 3.2. *Let X and Y be one-sided shift spaces and let $\Psi : O_X \to O_Y$ be a *-isomorphism satisfying $\Psi(C(X)) = C(Y)$ and $\Psi \circ \gamma^X = \gamma^Y \circ \Psi$. Then $\Psi(D_X) = D_Y$.*

PROOF. This follows from the observation that $D_X = C(X)' \cap \mathcal{F}_X$ and $D_Y = C(Y)' \cap \mathcal{F}_Y$. □

REMARK 3.3. Let X be any strictly sofic one-sided shift and let $Y = \tilde{X}$ be its cover. Then Y is (conjugate to) a shift of finite type so $D_Y = C(Y)$ but $D_X = C(Y) \ncong C(X)$. The identity map is a *-isomorphism $O_X \to O_Y$ with sends $D_X$onto $D_Y = C(Y)$, but there is no *-isomorphism $\Psi : O_X \to O_Y$ which satisfies $\Psi(C(X)) = C(Y)$.

Below, we give a stabilized version of Theorem 3.1. Consider the product $\tilde{X} \times \mathbb{N} \times \mathbb{T}$ equipped with the equivalence relation $\approx$ defined by $(\tilde{x}, m_1, z) \approx (\tilde{y}, m_2, w)$ if and only if $\tilde{x} = \tilde{y}$ and $m_1 = m_2$ and $z^n = w^n$ for all $n \in Iso(\tilde{x})$. The spaces $\tilde{X} \times \mathbb{N} \times \mathbb{T}/\approx$ and $(\tilde{X} \times \mathbb{T}/\sim) \times \mathbb{N}$ are now homeomorphic. An argument similar to the above then yields the following result.

COROLLARY 3.4. *Let X and Y be one-sided shift spaces with dense sets of aperiodic points and let $\Psi : O_X \otimes \mathbb{K} \to O_Y \otimes \mathbb{K}$ be a *-isomorphism satisfying $\Psi(C(X) \otimes c_0) = C(Y) \otimes c_0$. Then $\Psi(D_X \otimes c_0) = D_Y \otimes c_0$.*

References

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