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When every shift spaces are flow equivalent?

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Abstract. This paper will show that all shift spaces are flow equivalent to shifts of arbitrarily small entropy.

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1. Introduction

Let $\mathcal{A}$ be an alphabet. A word over $\mathcal{A}$ is a finite sequence over $\mathcal{A}$, written $w = w_1 ... w_n$ for $w_i \in \mathcal{A}$. We say that w has length $|w| = n$ and let the empty word of length 0 be denoted ϵ. If $w = w_1 .... w_n$ and $v = v_1 ... v_m$ are words over $\mathcal{A}$ then $wv = w_1 ... w_n v_1 ... v_m$ denotes their concatenation, and for $k \in \mathbb{N}$, w^k is the concatenation of k copies of w [2,5]. For a point $x = (x_i)_{i \in \mathbb{Z}}$ of the full shift $\mathcal{A}^{\mathbb{Z}}$, we let $x_{[i;j]}$; $i \leq j$, signify the word $w = x_i ... x_j$ and say that w *occurs* in x. Similarly, for a word $w = w_1 ... w_n$ over $\mathcal{A}$ and $i; j \in \mathbb{N}$ with $1 \leq i \leq j \leq n$, we denote by $w_{[i;j]}$ the subword $u = w_i ... w_j$ of w and say that u is a factor of w.
The canonical way of defining a shift space combinatorially is by the words that do not occur in any of its points. Let $\mathcal{A}$ be a finite alphabet, $\mathcal{F}$ a set of words over $\mathcal{A}$, and $X_{\mathcal{F}}$ the set of points $x \in \mathcal{A}^{\mathbb{Z}}$ such that no word of $\mathcal{F}$ occurs in x. The set $\mathcal{F}$ is called a set of *forbidden words* for $X_{\mathcal{F}}$.

Definition 1.1. Let $\mathcal{A}$ be an alphabet. A set $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is a *shift space* if there is a set of forbidden words $\mathcal{F}$ over $\mathcal{A}$ such that $X = X_{\mathcal{F}}$. If an arbitrary shift space X is given, we denote its alphabet by $\mathcal{A}(X)$, and the shift map restricted to X by σ_X.

We recall that for some alphabet $\mathcal{A}$, a shift space X over $\mathcal{A}$ is a compact, shiftinvariant subset of $\mathcal{A}^{\mathbb{Z}}$.

Definition 1.2. [1,2] Let X be a shift space and define an equivalence relation $\sim$ on $X \times \mathbb{R}$ generated by $(x; t+1) \sim (\sigma_X(x); t)$. Giving $X \times \mathbb{R}$ the product topology we let the *suspension flow* of X be given by the quotient space

$$SX = X \times \mathbb{R} / \sim \tag{1}$$

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We denote by $[x; t]$ the equivalence class in SX of $(x; t) \in X \times \mathbb{R}$.
A flow equivalence is a homeomorphism between the suspension flows of two shift spaces that preserves direction in $\mathbb{R}$.

Definition 1.3. [6] Let X and Y be shift spaces and SX and SY their suspension flows. A homeomorphism $\Phi : SX \to SY$ is a *flow equivalence* if for each $[x; t] \in SX$ there is a monotonically increasing function $\phi_{[x;t]} : \mathbb{R} \to \mathbb{R}$ such that $\Phi([x; t]) = [y; t']$ implies $\Phi([x; t + r]) = [y; t' + \phi_{[x;t]}(r)]$. If such a homeomorphism exists we say that X and Y are flow equivalent and write $X \sim_{FE} Y$.

Entropy describes the information density or complexity of a shift space by the asymptotic number of words of a given length.

Definition 1.4. Let X be a shift space. Then the entropy of X is given by

$$h(X) = lim_{n\to\infty} \frac{1}{n} log|B_n(X)|, \tag{2}$$

where *log* is the base 2 logarithm.

The limit exists (see for instance Lind and Marcus [[4], Prop. 4.1.8]), so the entropy is always well-defined. Entropy can be said to describe the information density of a shift space in the sense that if $h(X) = t > 0$ for some shift space X, then there are roughly 2^{tn} words of length n in X. A very intuitive example of entropy is that of the full shift.

Example 1.5. Let $X = X_{[r]}$ be the full r-shift. Then $|Bn(X)| = r^n$, so

$$h(X) = lim_{n\to\infty} \frac{1}{n} log|B_n(X)| = lim_{n\to\infty} \frac{n}{n} logr.$$

2. Main Section

This section will show that all shift spaces are flow equivalent to shifts of arbitrarily small entropy. First of all, we prove the already known result that having entropy zero is an invariant under flow equivalence.

Theorem 2.1. *Let X and Y be shift spaces with $X \sim_{FE} Y$. Then $h(X) = 0$ if and only if $h(Y) = 0$.*

Proof. Let X be a shift space. Since entropy is invariant under conjugacy and flow equivalence is generated by conjugacy and symbol expansion, we only need to show that for some shift space X and some shift space $Y = X^{a\mapsto a\lozenge}$ obtained by a symbol expansion of X, we have $h(X) = 0$ if and only if $h(Y) = 0$.
First, for $u_1; u_2 \in B_n(Y)$, it holds that $u_1^{\lozenge\mapsto\epsilon}$ is a prefix of $u_2^{\lozenge\mapsto\epsilon}$ if and only if u_2 can be achieved by adding and removing $\lozenge$ at the ends of u_1. This can be done in maximally of two different ways (e.g. if $u_1 = \lozenge u_1'$, where u_1' does not end in a $\lozenge$, then $u_2 = u_1$ or $u_2 = u_1'\lozenge$ are the two possibilities), so for every $w \in B_n(X)$ there is at most two words $u \in B_n(Y)$ such that $u^{\lozenge\mapsto\epsilon}$ is a prefix of w, and for every $u \in B_n(Y)$, $u^{\lozenge\mapsto\epsilon}$ is a prefix of some $w \in B_n(X)$. Thus, $2|B_n(X)| \geq |B_n(Y)|$, which yields

$$h(X) = lim_{n\to\infty} \frac{1}{n} log2|B_n(X)| \geq lim_{n\to\infty} \frac{1}{n} log|B_n(Y)| = h(Y). \tag{3}$$

Second, for two different word $w_1; w_2 \in B_n(X)$, none of the words $w_1^{a\mapsto a\lozenge}$ and $w_2^{a\mapsto a\lozenge}$ can be a prefix of the other, and for every $w \in B_n(X)$ there is at least one word $u \in B_{2n}(Y)$, which has $w^{a\mapsto a\Diamond}$ as a prefix. So $B_{2n}(Y) \geq B_n(X)$, and we can make the estimate

$$h(Y) = lim_{n\to\infty}\frac{1}{2n}log|B_{2n}(Y)| \geq lim_{n\to\infty}\frac{1}{2n}log|B_n(X)| = \frac{1}{2}h(X). \tag{4}$$

Thus, $h(X) \geq h(Y) \geq \frac{1}{2}h(X)$ and the result follows. □

PROPOSITION 2.2. *(Johansen [3]) Let X be a shift space and $a, b \in \mathcal{A}(X)$ with $a \neq b$. Then $X \sim_{FE} X^{a\mapsto ab}$.*

Moving on to shift spaces with non-zero entropy, we need a procedure that given a shift space can produce shift spaces flow equivalent to it of arbitrarily small entropy.

THEOREM 2.3. *Let X be a shift space and $n \in \mathbb{N}$. Then there exists $Y \sim_{FE} X$ with $h(Y) = \frac{1}{n}h(X)$.*

PROOF. The case $n = 1$ is trivially true, so assume that $n > 1$. Let $A = e_1, e_2, ..., e_m$ be the alphabet of X, $\Diamond \notin \mathcal{A}$, and $w = \Diamond^{n-1}$. Further, set $X_0 = X$ and consider the series of symbol expansions

$$X_i = X_{i-1}^{e_i\mapsto e_i w}, 1 \leq i \leq m.$$

Now,$X \sim_{FE} X_m$ by repeated use of Proposition 2.2, and for every $s \in \mathbb{N}$ the words of X_m of length ns can be described by

$$B_{ns}(X_m) = \{\Diamond^k f_1 w f_2 w ... w f_s \Diamond^{n-1-k} \mid 0 \leq k \leq n-1 \ and \ f_1 f_2 ... f_s \in B_s(X)\}.$$

So, noting that $|B_{ns}(X_m)| = n|B_s(X)|$, we find that

$$\frac{1}{n}h(X) = \frac{1}{n}lim_{s\to\infty}\frac{1}{s}log|B_s(X)| = lim_{s\to\infty}\frac{1}{ns}log\frac{1}{n}|B_{ns}(X_m)| = h(X_m).$$

□

The main result of the section now follows easily.

COROLLARY 2.4. *Any shift space X is flow equivalent to shifts of arbitrarily small entropy.*

PROOF. Follows directly from Theorem 2.1 □

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