Hypergroups and Lie hypergroups

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Abstract. Using the action of a Lie group on a hypergroup, the notion of Lie hypergroup is defined. It is proved that tangent space of a Lie hypergroup is a hypergroup and that a differentiable map between two Lie hypergroup is good homomorphism if and only if its differential map is a good homomorphism.

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1. Introduction

The concept of hypergroup arose originally as a generalization of the concept of abstract group. This concept was first introduced by Marty in 1935 [5]. Furthermore, some surveys and papers such as [1,5,8] were published in the field of hypergroup and its applications. M. R Molaei at.el studied semi hypergroups and their properties in [3,4,6]. In this paper, we introduce Lie hypergroup from a geometric point of view by using the action of a Lie group on a hypergroup. Some properties of quotient hypergroups are found. Indeed, using of the action of a Lie group on a hypergroup Lie hypergroup is introduced and some basic properties are given. It is proved that if left transformation on a Lie hypergroup be homomorphism then the transformation on its associated Lie hypergroup is homomorphism.

As follows, some basic notions and examples are reviewed.

Let P be a non-empty set and $p^*(P)$ be the set of all non empty subsets of P. A hyperoperation on P is a map $\circ : P \times P \to p^*(P)$ [2]. The ordered pair $(P, \circ)$ is called a *hypergroupoid*. If A and B are two non empty subsets of P and $x \in P$, then

$A \circ B = \cup_{a\in A, b\in B} a \circ b$, $x \circ A = \{x\} \circ A$ and $A \circ x = A \circ \{x\}$.

The hypergroupoid $(P, \circ)$ with the following properties is called a *hypergroup*.

1) $a \circ (b \circ c) = (a \circ b) \circ c$, for all $a, b, c \in P$,

2) $a \circ P = P \circ a = P$, for all $a \in P$.

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If there is an $e \in P$ such that $a \in a \circ e \cap e \circ a$, for all $a \in P$, then e is called *identity*. Let P be a hypergroup with at least one identity then an element $a^{-1} \in P$ is *inverse* of $a \in P$ if $e \in a \circ a^{-1} \cap a^{-1} \circ a$.
A hypergroup is called *regular* if it has at least one identity and for any element has at least one inverse.

Remark 1.1. If every element of p has inverse, so $(p_1 \circ p_2)^{-1} = p_2^{-1} \circ p_1^{-1}$ for all $p_1, p_2 \in P$.

Definition 1.2. [1] A subset S of a hypergroup $(P, \circ)$ is called *sub hypergroup* if it satisfies the following properties:
i) $a \circ b \subseteq S$ for all $a, b \in S$.
ii) $a \circ S = S \circ a = S$ for every $a \in S$.

Example 1.3. (affine join space) Let V be a vector space. Define hyperoperation $\circ : V \times V \longrightarrow P^*(V)$ which $a \circ b = \{\lambda a + \mu b \ : \ \lambda, \mu > 0 \ , \lambda + \mu = 1 \ \}$ for all $a, b \in V$. We can see easily V with this hyperoperation is a hypergroup and every subspace of V is a sub hypergoup.

A mapping f from a hypergroup $(P_1, \circ)$ to a hypergroup $(P_2, *)$ is called a
1) *homomorphism* if for all $x, y \in P_1$, $f(x \circ y) \subseteq f(x) \circ f(y)$.
2) *good homomorphism* if for all $x, y \in P_1$, $f(x \circ y) = f(x) * f(y)$.
Let (T, τ) be a topological space. Then, the family U consisting of all sets $S_V = \{U \in p^*(T) | U \subseteq V, U \in \tau\}$ is a basis for a topology on $p^*(T)$. This topology is denoted by τ^*. [9]

2. Lie hypergroup

In this section we define Lie hypergroup. Let $(P, \circ)$ be a hypergroup, G be a Lie group and G acts on P to the right by φ. We recall that $G_{p_0} = \{g \in G : \varphi(p_0, g) = p_0\}$ is called the *stabilizer* of action for $p_0 \in P$. If P be a manifold then $G/G_{p_0} \cong P$ [7]. Consider $f : P \longrightarrow G/G_{p_0}$ be the isomorphism function between hypergroup P and Lie hypergroup $\frac{G}{G_{p_0}}$. $f^{-1} : \frac{G}{G_{p_0}} \longrightarrow P$ is defined by $f^{-1}(gG_{p_0}) = \varphi(g, p_0)$. This isomorphism induces the following hyperoperation on G/G_{p_0}.

$$\circ' : G/G_{p_0} \times G/G_{p_0} \longrightarrow p^*(G/G_{p_0})$$

where $gG_{p_0} \circ' g'G_{p_0} = f(p \circ p')$ which $f(p) = gG_{p_0}$ and $f(p') = g'G_{p_0}$.
The identity set is $e(G/G_{p_0}) = \{f(e) \ : \ e \in e(P) \ \}$. If the hypergroup P is invertible, then G/G_{p_0} is invertible and the set of all inverses of gG_{p_0} in G/G_{p_0} is $i(gG_{p_0}) = \{g'G_{p_0} : f(e) \in gG_{p_0} \circ' g'G_{p_0} \cap g'G_{p_0} \circ' gG_{p_0} \ \}$. Hence, G/G_{p_0} is a hypergroup which we call Lie hypergroup.

Example 2.1. $O(3)$, the set of orthogonal 3×3 matrices, acts on S^2 in the following way:
$\varphi(A, x) = Ax$, for $A \in O(3)$ and $x \in S^2$. $O(3)/O(2) \cong S^2$ [7]. Using example 1.3 $O(3)/O(2)$ is a Lie hypergroup.
Let us investigate some basic properties of Lie hypergroups. actually, Theses are generalization of Lie group properties.

Remark 2.2. If P' is a sub hypergroup of P then $f(P')$ is a sub hypergroup of G/G_{p_0}.

THEOREM 2.3. *Let G be a Lie group and $p_0 \in P$ be a fixed element of hypergroup P. If $p_0 \in p_0 \circ p_0$ and $\varphi(g_1, p_0) \circ \varphi(g_2, p_0) = \varphi(g_1g_2, p_0 \circ p_0)$ for all $g_1, g_2 \in G$. Then $g_1g_2G_{p_0} \in g_1G_{p_0} \circ' g_2G_{p_0}$.*

PROOF. Let $p_1 = \varphi(g_1, p_0)$ and $p_2 = \varphi(g_2, p_0)$. By assumptions

$$p_1 \circ p_2 = \varphi(g_1, p_0) \circ \varphi(g_2, p_0) = \varphi(g_1g_2, p_0 \circ p_0).$$

Hence, $f^{-1}(g_1g_2G_{p_0}) = \varphi(g_1g_2, p_0) \in p_1 \circ p_2$. So, $g_1g_2G_{p_0} \in f(p_1 \circ p_2) = g_1G_{p_0} \circ' g_2G_{p_0}$. proof

THEOREM 2.4. *Let $(P_1, \circ_1)$ and $(P_2, \circ_2)$ be hypergroups. Consider $\varphi_1 : P_1 \times G \longrightarrow P_1$ and $\varphi_2 : P_2 \times G \longrightarrow P_2$ are good homomorphism actions. Consider $p_1 \in P_1$ and $p_2 \in P_2$ are arbitrary and fixed points. If $\psi_0 : G/G_{p_1} \longrightarrow G/G_{p_2}$ is a good homomorphism then there is a map $\psi : P_1 \longrightarrow P_2$ such that the following diagram commutes and ψ is a good homomorphism. Where $f : P_1 \longrightarrow G/G_{p_1}$ and $g : P_2 \longrightarrow G/G_{p_2}$ are isomorphisms which are correspond the actions φ_1 and φ_2.*

PROOF. Using assumptions $\psi(\omega) = g^{-1} \circ \psi_0 \circ f(\omega) = \varphi_2(p_2, \psi_0(f(\omega)))$ for all $\omega \in P_1$. Therefore,
$\psi(\omega_1) \circ_2 \psi(\omega_2) = \varphi_2(p_2, \psi_0(f(\omega_1))) \circ_2 \varphi_2(p_2, \psi_0(f(\omega_2))) = \varphi_2(p_2, \psi_0(f(\omega_1)) \circ_2' \psi_0(f(\omega_2))) = \varphi_2(p_2, \psi_0(f(\omega_1)) \circ_2' f(\omega_2))) = \varphi_2(p_2, \psi_0(f(\omega_1 \circ_1 \omega_2)) = g^{-1} \circ \psi_0 \circ f(\omega_1 \circ_1 \omega_2) = \psi(\omega_1 \circ_1 \omega_2)$. □

As follows, tangent space of a hypergroup is introduced as a hypergroup. Consider T_pP is tangent space on the manifold P at point p and $TP = \cup_{p \in P} T_pP$ is tangent bundle. In addition, suppose that $(P, \circ)$ be a hypergroup. Let $v_1 \in T_{p_1}P$ and $v_2 \in T_{p_2}P$. Define the hyperoperation $\# : TP \times TP \longrightarrow p^*(TP)$ where $v_1 \# v_2 = \{v \quad : \quad v \in T_pP \quad p \in p_1 \circ p_2 \}$.

THEOREM 2.5. *$(TP, \#)$ is a hypergroup.*

PROOF. Let v_1, v_2 and v_3 are arbitrary tangent vectors.
$v_1 \#(v_2 \# v_3) = \{v_1 \# v \quad : \quad v \in T_pP \quad p \in p_1 \circ p_2 \}$
$= \{v' \quad : \quad v' \in T_{p'}P \quad p' \in p_1 \circ p \} = \{v' \quad : \quad v' \in T_{p'}P \quad p' \in p_1 \circ (p_2 \circ p_3) \}$
$= \{v' \quad : \quad v' \in T_{p'}P \quad p' \in (p_1 \circ p_2) \circ p_3 \} = (v_1 \# v_2) \# v_3$.
Also for $v \in T_pP$, we have
$v \# TP = \{v' \quad : v' \in v \# v_1, \ v_1 \in T_{p_1}P \quad \exists p_1 \in P\} = \{v' \quad : \quad v' \in T_{p'}P \quad p' \in p \circ p_1\} = TP$.
Thus, $(TP, \#)$ is a hypergroup.
This implies that the tangent bundle of a Lie hypergroup is a hypergroup. Let $(P_1, \circ_1)$ and $(P_2, \circ_2)$ be hypergroups which are manifold too. By previous theorem $(TP_1, \#)$ and $(TP_2, \#')$ are hypergroups. □

THEOREM 2.6. *If $\psi : P_1 \longrightarrow P_2$ is a differentiable function. Then, ψ is good homomorphism if and only if $d\psi : TP_1 \longrightarrow TP_2$ is good homomorphism.*

PROOF. Let $\omega_1, \omega_2 \in P_1$ be arbitrary and $v_1 \in T_{\omega_1}P_1$ and $v_2 \in T_{\omega_2}P_1$. Firsly, consider ψ is a good homomorphism. Let $v \in v_1 \# v_2$, $d\psi(v) \in d\psi(v_1 \# v_2)$ such that $v \in T_\omega P_1$ and $\omega \in \omega_1 \circ_1 \omega_2$. Hence, $d\psi(v) \in T_{\psi(\omega)}P_2$ and $\psi(\omega) \in \psi(\omega_1 \circ_1 \omega_2) = \psi(\omega_1) \circ_2 \psi(\omega_2)$. On the other hand,

$$d\psi(v_1) \#' d\psi(v_2) = \{v' \quad : \quad v' \in T_{\omega'}P_2 \quad , \ \omega' \in \psi(\omega_1) \circ_2 \psi(\omega_2)\}.$$

Thus, $d\psi(v) \in d\psi(v_1) \#' d\psi(v_2)$.
Conversely, if $v \in d\psi(v_1) \#' d\psi(v_2)$ then $v \in T_{\psi(\omega)}P_2$. Since, ψ is onto map so, there is

$v' \in T_\omega P_1$ such that $v' = d\psi(v)$ and $v \in v_1 \# v_2$. Hence, $v \in d\psi(v_1 \# v_2)$.
Secondly, consider $d\psi$ is good homomorphism. Let $\omega' \in \psi(\omega_1) \circ_2 \psi(\omega_2)$ and $v' \in T_{\omega'} P_2$ so, $v' \in d\psi(v_1)\#' d\psi(v_2) = d\psi(v_1 \# v_2)$ where $v_1 \in T_{\omega_1} P_1$ and $v_2 \in T_{\omega_2} P_1$. Therefore, there is $v \in v_1 \# v_2$ such that $v' = d\psi(v)$ and there exists an $\omega \in \omega_1 \circ_1 \omega_2$ such that $v' \in T_{\psi(\omega)} P_2$. Hence, $\omega' = \psi(\omega) \in \psi(\omega_1 \circ_1 \omega_2)$.
Conversely, let $\omega \in \omega_1 \circ_1 \omega_2$ and $\omega' = \psi(\omega)$. If $v \in T_\omega P_1$ then $d\psi(v) \in T_{\omega'} P_2$. Since, $d\psi$ is good homomorphism, $d\psi(v) \in T_{\psi(\omega)} P_2$. Therefore, $\psi(\omega) \in \psi(\omega_1) \circ_2 \psi(\omega_2)$. □

Remark 2.7. Let $l_{gG_{p_0}} : G/G_{p_0} \longrightarrow G/G_{p_0}$ be left transformation on G/G_{p_0} where $l_{gG_{p_0}}(g'G_{p_0}) = gg'G_{p_0}$. Then there is a transformation $l_g : P \longrightarrow P$ such that $f \circ l_g = l_{gG_{p_0}} \circ f$.

Theorem 2.8. *If $l_{gG_{p_0}} : G/G_{p_0} \longrightarrow G/G_{p_0}$ is a homomorphism left transformation. Then $l_g : P \longrightarrow P$ is homomorphism.*

Proof. . Let $p_1, p_2 \in P$ be arbitrary and there are $g_1, g_2 \in G$ such that $f(p_1) = g_1 G_{p_0}$ and $f(p_2) = g_2 G_{p_0}$. Then
$l_g(p_1 \circ p_2) = \{l_g(p) \ : \ p \in p_1 \circ p_2 \} = \{f^{-1} \circ l_{gG_{p_0}}(g'G_{p_0}) \ : \ f(p) = g'G_{p_0}, p \in p_1 \circ p_2 \} = \{f^{-1}(gg'G_{p_0}) \ : \ g'G_{p_0} \in g_1 G_{p_0} \circ' g_2 G_{p_0} \} = \{f^{-1}(l_{gG_{p_0}}(g'G_{p_0})) \ : \ g'G_{p_0} \in g_1 G_{p_0} \circ' g_2 G_{p_0} \} \subseteq f^{-1}(l_{gG_{p_0}}(g_1 G_{p_0}) \circ' l_{gG_{p_0}}(g_2 G_{p_0})) = f^{-1}(gg_1 G_{p_0} \circ' gg_2 G_{p_0}) = f^{-1}(g_1 g G_{p_0}) \circ f^{-1}(gg_2 G_{p_0}) = (f^{-1} \circ l_{gG_{p_0}} \circ f)(p_1) \circ (f^{-1} \circ l_{gG_{p_0}} \circ f)(p_2) = l_g(p_1) \circ l_g(p_2)$. □

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