A note on causal conditions fail along a null geodesic

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Abstract. It is known for some of the causality conditions that they can't fail at a single isolated point. Recetlely, it is shown that if causal continuity or stable causality fail at a point p then there is a null geodesic segment containing p at every point of which the condition fails. In this paper, we show that if causal simplicity fails at a point p of a reflecting spacetime M then there exists a future or past inextendible maximal null geodesics with endpoint p at every point of which causal simplicity fails.

AMS Mathematics Subject Classification [2020]: 83Cxx, 53C50.

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1. Introduction

In the theory of General Relativity, a space-time (M, g) is a connected C^∞ Hausdorff manifold of dimension two or greater which has a countable basis, a Lorentzian metric g of signature $(-, +, ..., +)$ and a time orientation. The metric g determines the causal structure of the space-time based on which causality conditions are defined. Causality conditions are important in determining how physical a space-time is and proving mathematical theorems about its global structure.

We say that a vector $v \in T_pM$ is timelike if $g_p(v, v) < 0$, causal if $g_p(v, v) \leq 0$, null if $g_p(v, v) = 0$ and spacelike if $g_p(v, v) > 0$. A smooth curve is timelike (future pointing) if its tangent vector is everywhere timelike (future pointing). When speaking about future pointing curves, we usually omit future pointing and simply write causal or timelike curve. Causal and null, future or past pointing and space-like curves are defined similarly. Suppose $p, q \in M$. q is in the chronological future of p, written $q \in I^+(p)$ or $p \ll q$, if there is a timelike future pointing curve $\gamma : [0, 1] \to M$ with $\gamma(0) = p$, and $\gamma(1) = q$; similarly, q is in the causal future of p, written $q \in J^+(p)$ or $p \prec q$, if there is a future pointing causal curve from p to q. For any point, p, $I^\pm(p)$ is open; but $J^\pm(p)$ need not, in general, be closed. $J^\pm(p)$ is, however, always a subset of the closure of $I^\pm(p)$.

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The set of all the Lorentzian metrics on M is denoted by $Lor(M)$. The fine C^0 topology on $Lor(M)$ is defined using a fixed locally finite countable covering $B = \{B_i\}$ of M by coordinate neighborhoods with the property that the closure of each B_i lies in a coordinate chart of M. Let $\delta : M \longrightarrow (0, \infty)$ be a continuous function. Then $g_1, g_2 \in Lor(M)$ are said to be σ close in the C^0 topology, if for each $p \in M$ all of the corresponding coefficients of the two metrics are $\sigma(p)$ close at p when calculated in the fixed coordinates of all $B_i \in B$ which contain p. For all $g, h \in Lor(M)$, $h > g$ iff $p \prec q$ in space-time (M, g) implies $p \ll q$ in space-time (M, h).

To be more careful, it is useful to remind the following conditions. [2,3] A space-time M is:

- Causal if there is no non-degenerate causal curve which starts and ends at the same point. If M is causal at all points it's simply called causal.
- Strongly causal at p if p has arbitrarily small neighborhoods which every causal curve intersects in a single component.
- Stably causal if there is a fine C^0 neighborhood $U(g)$ of g in Lor(M) such that each $h \in U(g)$ is causal (equivalently there exists some causal $h \in Lor(M)$ with $g < h$).
- Causally continuous at a point p if for each compact set K in the exterior of $I^+(p)$ there exists some neighborhood $U(p)$ of p such that for each $q \in U(p)$, K is in the exterior of $I^+(q)$.
- Causally simple at p if it is strongly causal and $J^{\pm}(p)$ is closed.
- Globally hyperbolic if it is strongly causal and $J^+(p) \cap J^-(q)$ is compact for all $p, q \in M$.

2. Main results

Some causality conditions are defined pointwisely i.e. they either hold or not *at individual points*. A natural question is whether the failure of a pointwise causality condition at a point of space-time, implies the failure of the condition at some other points.

As a result of Proposition 4.29 and Theorem 3.31 in [4], one can deduce that the failure of future or past distinction and strong causality conditions at some point $p \in M$ imply the failure of them at all points of a null geodesic segment containing p. It is also not hard to show that chornologicality and causality have this property too. In Ref. [1], it is shown that causal continuity and an equivalent pointwise defition of stable causality also have this property i.e. if they fail at a point p there is a null geodesic segment containing p along which the conditions fail.

Theorem 2.1. *[1] If causal continuity fails at a point p then there is a future endless null geodesic γ with past point p at every point of which causal continuity fails.*

Theorem 2.2. *[1] Let S^c be the set of points atwhich stable causality fails. If stable causality fails at p then at least one of the following holds:*
a) $p \in int(S^c)$;
b) p is a non-endpoint point on a future endless null geodesic on ∂S^c;
c) p is a non-endpoint point on a past endless null geodesic on ∂S^c;
d) p is the endpoint of a future endless and a past endless null geodesic on ∂S^c;
e) p lies on an endless null geodesic on ∂S^c at every point of which stable causality fails.

So, the above theorem shows that when stable causality fails at a point p, there is a null geodesic segment containing p along which the condition fails. But, the case of causal simplicity is a challenging problem and remains a conjecture [1]. Now, by the following theorem, we prove it.

THEOREM 2.3. *If causal simplicity fails at a point p of a reflecting spacetime M then there exists a future or past inextendible maximal null geodesics with endpoint p at every point of which causal simplicity fails.*

PROOF. Let M not be causally simple. Therefore, there exists a point p that $J^+(p)$ or $J^-(p)$ is not closed. We show that if $J^+(p)$ ($J^-(p)$) is not closed then there exists a null geodesic segment with past (future) endpoint p at every point of which causal simplicity fails. Since $J^+(p)$ is not closed, there is a point $r \in \partial J^+(p) \setminus J^+(p)$ and a sequence of points $r_n \in I^+(p)$ which $r_n \longrightarrow r$ and also there is a sequence of causal curves γ_n from p to r_n . Let $U(p)$ be a strictly convex normal neighborhood of p such that $\partial U(p)$ is compact. So, γ_n intersects $\partial U(p)$ in p'_n. Assume p'_n converge to $p' \in \partial U(p)$. So, there exists a causal geodesic pp'. There exists two possibilities:

Case 1: The geodesic pp' is timelike.
In this case, $p \in I^-(p')$ and any future null geodesic with past endpoint p has a segment in $I^-(p')$ such that for every point s of this segment, $r \in \partial J^+(s) \setminus J^+(s)$ and $J^+(s)$ is not closed.
Case 2: The geodesic pp' is null.
In this case, we show that causal simplicity fails at every points of the null geodesic pp'. By the reflectivity of M, $r \in \partial J^+(p') \setminus J^+(p')$ and we conclude that $J^+(p')$ is not closed. Now, for every point q on the null geodesic pp' we have $r \in \partial J^+(p') \subseteq \overline{J^+(q)}$ and $r \notin J^+(q) \subseteq J^+(p)$ and therefore, $J^+(q)$ is not closed.

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