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# Schouten and Vranceanu connections on metallic manifold

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Abstract. In this paper, we introduce two linear connections, that are called Schouten and Vranceanu connections, on a metallic Riemannian manifold and study the notion of parallelism for distributions derived frome metallic structure with respect to the Schouten and Vranceanu connections.

**Keywords:** metallic structure, Schouten connection, Vranceanu connection, parallelism, half parallesim.

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## 1. Introduction

In 1999, the concept of the metallic ratio was introduced by Vera W. de Spinadel [4]. Recently, many researchers studied metallic structures [2, 3].
Now, we recall some necessary notions.
Let $M$ be a $C^\infty$ manifold, A tensor $\Phi$ of type $(1,1)$ is said to be a metallic structure if

$$J^2 = pJ + qI.$$

A $C^\infty$ manifold $M$ eqquiped with a metallic structure $J$ is called a metallic manifold and denoted by $(M, J)$ [2]. The solutions of the equation $x^2 - px - q = 0$ are named the metallic means family and denoted by

$$\sigma_+ = \frac{p + \sqrt{p^2+4q}}{2}, \quad \sigma_- = \frac{p - \sqrt{p^2+4q}}{2} \tag{1}$$

The projections operator with respect to metallic structure $J$ are as follows:

$$\mathcal{P} = \frac{-1}{\sqrt{p^2+4q}} J + \frac{\sigma_+}{\sqrt{p^2+4q}} I,$$

$$\mathcal{P}' = \frac{1}{\sqrt{p^2+4q}} J - \frac{\sigma_-}{\sqrt{p^2+4q}} I. \tag{2}$$

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These maps satisfy in the following relations:

$$\mathcal{P}^2 = \mathcal{P},\ \mathcal{P}'^2 = \mathcal{P}',\ \mathcal{P} + \mathcal{P}' = I.$$

Also, we have $J = \sigma_+\mathcal{P} + \sigma_-\mathcal{P}'$, where $J$ is metallic structure.

The coresponding distributions with respect to $\mathcal{P}$ and $\mathcal{P}'$ are as follows:

$$D_p = \{X_p \in T_pM,\ JX_p = \sigma_+X_p\},\ D = \bigcup_p D_p,$$

$$D'_p = \{X_p \in T_pM,\ JX_p = \sigma_-X_p\},\ D' = \bigcup_p D'_p, \tag{3}$$

so $TM = D \oplus D'$.

## 2. Main Results

In this section, we investigate the notion of parallelism for the distributions $D$ and $D'$, defined in (3), with respect to the Schouten and Vranceanu connections.

Definition 2.1. The Schouten and Vranceanu connections with respect to metallic structure are defined as follows [1]:

$$\nabla^S_X Y = \mathcal{P}\nabla_X\mathcal{P}Y + \mathcal{P}'\nabla_X\mathcal{P}'Y,$$

$$\nabla^V_X Y = \mathcal{P}\nabla_{\mathcal{P}X}\mathcal{P}Y + \mathcal{P}'\nabla_{\mathcal{P}'X}\mathcal{P}'Y + \mathcal{P}[\mathcal{P}'X, \mathcal{P}Y] + \mathcal{P}'[\mathcal{P}X, \mathcal{P}'Y].$$

Definition 2.2. Let $D$ be a distribution on $M$. $D$ is parallel with respect to linear connection $\nabla$ if the vector $\nabla_X Y \in \Gamma D$.

Proposition 2.3. *Let $(M, J)$ be a metallic Riemannian manifold and $D$ and $D'$ be the distributions with respect to $\mathcal{P}$ and $\mathcal{P}'$ defined in (2) and (3). The following statements are valid.*

1) *Both of the distributions $D$ and $D'$ are parallel with respect to the Schouten and Vranceanu connection.*
2) $\nabla^S_X\mathcal{P}Y = \mathcal{P}\nabla_X\mathcal{P}Y.$

*for all $X, Y \in TM$.*

Proof. 1) We must show that $\nabla^S_X Y, \nabla^V_X Y \in \Gamma(D)$. For this perpose we have if $Y \in \Gamma(D)$ then $\mathcal{P}'(Y) = 0$, so

$$\nabla^S_X Y = \mathcal{P}\nabla_X\mathcal{P}Y + \mathcal{P}'\nabla_X\mathcal{P}'Y = \mathcal{P}\nabla_X\mathcal{P}Y \in (\Gamma D).$$

and

$$\begin{aligned}\nabla^V_X Y &= \mathcal{P}\nabla_{\mathcal{P}X}\mathcal{P}Y + \mathcal{P}'\nabla_{\mathcal{P}'X}\mathcal{P}'Y + \mathcal{P}[\mathcal{P}'X, \mathcal{P}Y] + \mathcal{P}'[\mathcal{P}X, \mathcal{P}'Y] \\ &= \mathcal{P}\nabla_{\mathcal{P}X}\mathcal{P}Y + \mathcal{P}[\mathcal{P}'X, \mathcal{P}Y] \in \Gamma(D).\end{aligned}$$

Similarly, we can prove this for $D'$.

2) By a straightforward calculation we get

$$\begin{aligned}\nabla^S_X\mathcal{P}Y &= \mathcal{P}\nabla_X\mathcal{P}^2Y + \mathcal{P}'\nabla_X\mathcal{P}'\mathcal{P}Y \\ &= \mathcal{P}\nabla_X\mathcal{P}Y.\end{aligned} \tag{4}$$

□

Proposition 2.4. *Let $(M, J)$ be a metallic Riemannian manifold. Then $\nabla^S_X\mathcal{P} = 0$, where $\mathcal{P}$ is the projection map from $TM$ to $D$.*

PROOF. we have

$$\nabla^S_X \mathcal{P}Y = \mathcal{P}(\nabla^S_X Y) + (\nabla^S_X \mathcal{P})Y = \mathcal{P}(\mathcal{P}\nabla_X \mathcal{P}Y + \mathcal{P}'\nabla_X \mathcal{P}'Y) + (\nabla^S_X \mathcal{P})Y$$
$$= \mathcal{P}\nabla_X \mathcal{P}Y + (\nabla^S_X \mathcal{P})Y, \quad (5)$$

According to Eqs. (4) and (5), we obtained $(\nabla^S_X \mathcal{P})Y = 0$, for all $Y \in TM$, so $\nabla^S_X \mathcal{P} = 0$ □

PROPOSITION 2.5. *Let $(M, J)$ be a metallic Riemannian manifold. Then the metallic structure $J$ is parallel with respect to Schouten and Vranceanu connections.*

PROOF. we have $\nabla^S_X JY = \mathcal{P}\nabla_X \mathcal{P}JY + \mathcal{P}'\nabla_X \mathcal{P}'JY$. Since

$$\mathcal{P}J = J\mathcal{P} = \sigma_+ \mathcal{P},$$

and

$$\mathcal{P}'J = J\mathcal{P}' = \sigma_- \mathcal{P},$$

so we have

$$\nabla^S_X JY = \mathcal{P}\nabla_X \sigma_+ \mathcal{P}Y + \mathcal{P}'\nabla_X \sigma_- \mathcal{P}Y$$
$$= \sigma_+ \mathcal{P}\nabla_X \mathcal{P}Y + \sigma_- \mathcal{P}'\nabla_X \mathcal{P}Y. \quad (6)$$

on the other hand

$$\nabla^S_X JY = (\nabla^S_X J)Y + J(\nabla^S_X Y)$$
$$= (\nabla^S_X J)Y + J(\mathcal{P}\nabla_X \mathcal{P}Y + \mathcal{P}'\nabla_X \mathcal{P}'Y)$$
$$= (\nabla^S_X J)Y + \sigma_+ \mathcal{P}\nabla_X \mathcal{P}Y + \sigma_- \mathcal{P}'\nabla_X \mathcal{P}Y). \quad (7)$$

By Eqs. (6) and (7), we have $(\nabla^S_X J)Y = 0$, for all $Y \in TM$. So $\nabla^S_X J = 0$.

Similarly, we can show that $\nabla^V_X J = 0$. □

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