CONTACT EQUIVALENCE PROBLEM FOR THE GENERAL FORM OF BURGERS' EQUATIONS

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Abstract. The moving coframe method is applied to solve the local equivalence problem for the equation of the form $u_{xx} = u_t + Q(u)u_x$ in two independent variables under an action of the pseudo-group of contact transformations. The structure equations, invariants and equivalent condition of this equations are found.

Introduction

In this article we consider a local equivalence problem for the class of equations

$$u_{xx} = u_t + Q(u)u_x \tag{1}$$

under a contact transformation pseudo-group. Two equations are said to be equivalent if there exists a contact transformation mapping one equation to the other.We use Elie Cartan's method of equivalence, [1], in its form developed by Fels and Olver, [2, 3], to compute the Maurer - Cartan forms, the structure equations, the basic invariants, and the invariant derivatives for symmetry groups of equations from the class . All differential invariants are functions of the basic invariants and their invariant derivatives. Cartan's solution to the equivalence problem states that two equations are (locally) equivalent if and only if Cartan test's satisfied.

1. equivalence problem of differential equations

In this section we describe the local equivalence problem for differentials equations under the action of the pseudo group of contact transformations. Two equations are said to be equivalent if there exists a contact transformation which maps the equations to each other. We apply Elie Cartan's structure theory of Lie pseudo-groups to obtain necessary and sufficient conditions under which equivalence mappings can be found. This theory describes a Lie pseudo-group in terms of a set of invariant differential 1-forms called Maurer-Cartan forms. Expressions of exterior differentials of Maurer-Cartan forms in terms of the forms themselves yield Cartan structure equations for the pseudo-group. The Maurer-Cartan forms contain all information about the pseudo-group, in particular, they give basic invariants and operators of invariant differentiation and allow one to solve equivalence problems

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for submanifolds under the action of the pseudo-group.
As is shown in [4], the following differential 1-forms,

$$\begin{aligned}
\Theta^\alpha &= a^\alpha_\beta(du^\beta - u^\beta_{x^j}dx^j),\\
\Xi^i &= b^i_j dx^j + c^i_\beta \Theta^\beta,\\
\Sigma^\alpha_i &= a^\alpha_\beta B^i_j du^\beta_{x^j} + f^\alpha_{i\beta}\Theta^\beta + g^\alpha_{ij}\Xi^j
\end{aligned}$$

are Maurer-Cartan forms of $Cont(J^1(\pi))$. They are defined on $J^1(\pi)\times\mathcal{H}$, where $\mathcal{H} = (a^\alpha_\beta, b^i_j, c^i_\beta, f^\alpha_{i\beta}, g^\alpha_{ij}) \mid \alpha,\beta \in \{1,\ldots,q\}, i,j \in \{1,\ldots,n\}, det(a^\alpha_\beta).det(b^i_j) \neq 0, g^\alpha_{ij} = g^\alpha_{ji}$,$(B^i_j)$ is the inverse matrix for (b^i_j). They satisfy the structure equations

$$\begin{aligned}
d\Theta^\alpha &= \Phi^\alpha_\beta\wedge\Theta^\beta + \Xi^k\wedge\Sigma^\alpha_k,\\
d\Xi^i &= \Psi^i_k\wedge\Xi^k + \Pi^i_\gamma\wedge\Theta^\gamma,\\
d\Sigma^\alpha_i &= \Phi^\alpha_\gamma\wedge\Sigma^\gamma_i - \Psi^k_i\wedge\Sigma^\alpha_k + \Lambda^\alpha_{i\beta}\wedge\Theta^\beta + \Omega^\alpha_{ij}\wedge\Xi^j.
\end{aligned}$$

where the forms$\Phi^\alpha_\beta, \Psi^i_j, \Pi^i_\beta, \Lambda^\alpha_{i\beta}$ and Ω^α_{ij} depend on differentials of the coordinates of $\mathcal{H}$. Differential equations defines a submanifold $\mathcal{R}\subset J^1(\pi)$. The Maurer-Cartan forms for its symmetry pseudo-group Cont($\mathcal{R}$) can be found from restrictions $\theta^\alpha = \imath^*\Theta^\alpha, \xi^i = \imath^*\Xi^i$and $\sigma^\alpha_i = \imath^*\Sigma^\alpha_i$. where $\imath = \imath_0\times id : \mathcal{R}\times\mathcal{H}\longrightarrow J^1(\pi)\times\mathcal{H}$ with $\imath_0 : \mathcal{R}\longrightarrow J^1(\pi)$ defined by our differential equations. In order to compute the Maurer.Cartan forms for the symmetry pseudo-group, we implement Cartan's equivalence method. Firstly, the forms $\theta^\alpha, \xi^i, \sigma^\alpha_i$ are linearly dependent, i.e. there exists a nontrivial set of functions $U_\alpha, V_i, W^i_\alpha$ on $\mathcal{R}\times\mathcal{H}$ such that $U_\alpha\theta^\alpha + V_i\xi^i + W^i_\alpha\sigma^\alpha_i \equiv 0$. Setting these functions equal to some appropriate constants allows one to express a part of the coordinates of $\mathcal{H}$ as functions of the other coordinates of $\mathcal{R}\times\mathcal{H}$. Secondly, we substitute the obtained values into the forms $\phi^\alpha_\beta = \imath^*\Phi^\alpha_\beta$ and $\psi^i_k = \imath^*\psi^i_k$ coefficients of semi-basic forms ϕ^α_β at σ^γ_j, ξ^j, and the coefficients of semi-basic forms ψ^i_j at σ^γ_j are lifted invariants of $Cont(\mathcal{R})$. We set them equal to appropriate constants and get expressions for the next part of the coordinates of $\mathcal{H}$, as functions of the other coordinates of $\mathcal{R}\times\mathcal{H}$. Thirdly, we analyze the reduced structure equations

$$\begin{aligned}
d\theta^\alpha &= \phi^\alpha_\beta\wedge\theta^\beta + \xi^k\wedge\sigma^\alpha_k,\\
d\xi^i &= \psi^i_k\wedge\xi^k + \pi^i_\gamma\wedge\theta^\gamma,\\
d\sigma^\alpha_i &= \phi^\alpha_\gamma\wedge\sigma^\gamma_i - \psi^k_i\wedge\sigma^\alpha_k + \lambda^\alpha_{i\beta}\wedge\theta^\beta + \omega^\alpha_{ij}\wedge\xi^j.
\end{aligned}$$

If the essential torsion coefficients dependent on the group parameters appear, then we should normalize them to constants and find some new part of the group parameters, which, on being substituted into the reduced modified Maurer-Cartan forms, allows us to repeat the procedure of normalization. There are two possible results of this process. The first result, when the reduced lifted coframe appears to be involutive, outputs the desired set of invariant 1-forms which characterize the pseudo-group $Lie(\mathcal{R})$. In the second result, when the coframe is not involutive, we should apply the procedure of prolongation [[5]].

2. Structure of symmetry groups for general form of Burgers' equations

We apply the method described in the previous section to the class of equations (1).we take the equivalent system of first order

$$u_x = v, \qquad u_t = v_x + Q(u)v. \tag{2}$$

Denoting, $x = x_1, t = x_2, v = u_1, u = u_2, v_x = p_1^1, v_t = p_2^1, u_t = p_2^2, u_x = p_1^2$. We consider this system as a sub-bundle of the bundle $J^1(\varepsilon), \varepsilon = \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, with local coordinates $\{x_1, x_2, u_1, u_2, p_1^1, p_2^1, p_1^2, p_2^2\}$, where the embedding ι is defined by the equalities:

$$p_1^2 = u_1, \qquad p_1^1 = p_2^2 - Q(u_2)u_1. \tag{3}$$

The forms $\theta^\alpha = \iota^*\Theta^\alpha, \alpha \in \{1,2\}, \xi^i = \iota^*\Xi^i, i \in \{1,2\}$, are linearly dependent. The group parameters a_β^α, b_j^i must satisfy the conditions $det(a_\beta^\alpha) \neq 0, det(b_j^i) \neq 0$. linear dependence between the forms σ_i^α are

$$\sigma_1^2 = 0, \qquad \sigma_1^1 = \sigma_2^2 \tag{4}$$

Computing the linear dependence conditions (4) gives the following group parameters as a functions of other group parameters and the local coordinates $\{x_1, x_2, u_1, u_2, p_2^1, p_2^2\}$ of $\mathcal{R}$. In particular,

$$\begin{aligned}
&a_1^2 = 0, \qquad b_1^2 = 0, \qquad b_2^2 = \frac{b_1^1 a_2^2}{a_1^1},\\
&g_{12}^2 = -\frac{a_1^1(Qb_2^1u_1 + b_1^1p_2^1 - b_2^1p_2^2)}{(b_1^1)^3}, \qquad g_{11}^2 = \frac{a_2^2(Qu_1 - p_2^2)}{(b_1^1)^2},\\
&g_{11}^1 = \frac{\frac{dQ}{du_2}{u_1}^2a_1^1 - Q^2a_1^1u_1 + Qa_1^1p_2^2 + Qa_2^1u_1 - a_1^1p_2^1 - a_2^1p_2^2}{(b_1^1)^2},
\end{aligned} \tag{5}$$

The expression for $f_{12}^2, f_{11}^2, f_{21}^2, f_{22}^2, g_{22}^2$ is too long to be written out in full here. The analysis of the semi-basic modified Maurer-Cartan forms $\phi_\beta^\alpha, \psi_k^i, \pi_\gamma^i$ at the obtained values of the group parameters gives the following normalizations:

$$a_2^2 = a_1^1 b_1^1, \qquad c_1^2 = 0, \qquad c_2^2 = c_1^1. \tag{6}$$

The expression for f_{11}^1 is too long to be written out in full here.
The analysis of the structure equations gives the following normalizations:

$$\begin{aligned}
&c_2^1 = 0, \qquad c_1^1 = 0, \qquad a_2^1 = \frac{1}{2}\frac{a_1^1(Qb_1^1 - b_2^1)}{b_1^1},\\
&f_{12}^1 = \frac{1}{12}\frac{4(b_1^1)^4 f_{21}^1 + 8(\frac{dQ}{du_2})(b_1^1)^2u_1 - (Qb_1^1)^2 + 2Qb_1^1b_2^1 - (b_2^1)^2}{(b_1^1)^4},\\
&f_{22}^1 = \frac{1}{2}\frac{(\frac{dQ}{du_2})\left(2Qb_1^1u_1 - b_1^1p_2^2 - b_2^1u_1\right) - (\frac{d^2Q}{d{u_2}^2})b_1^1{u_1}^2}{(b_1^1)^4}.
\end{aligned} \tag{7}$$

Regarding the appearance of different derivatives of $Q(u)$ in the essential torsion coefficients and with respect to vanishing or non-vanishing of these derivatives and their effects on normalizations process, we have to impose some restrictions on the function $Q(u_2)$. As a result of these restrictions, the following cases arise.

Case-1:

After normalization (7), if Q is a constant then we have the following structure equations

$$\begin{aligned}
d\theta^1 &= \phi_1^1 \wedge \theta^1 - \frac{1}{2}\psi_2^1 \wedge \theta^2 + \xi^1 \wedge \sigma_2^2 + \xi^2 \wedge \sigma_2^1,\\
d\theta^2 &= \phi_1^1 \wedge \theta^2 + \psi_1^1 \wedge \theta^2 - \theta^1 \wedge \xi^1 + \xi^2 \wedge \sigma_2^2,\\
d\xi^1 &= \psi_1^1 \wedge \xi^1 + \psi_2^1 \wedge \xi^2,\\
d\xi^2 &= 2\psi_1^1 \wedge \xi^2,\\
d\sigma_2^1 &= \phi_1^1 \wedge \sigma_2^1 - 2\psi_1^1 \wedge \sigma_2^1 - \frac{3}{2}\psi_2^1 \wedge \sigma_2^2 + \lambda_{21}^1 \wedge \theta^1 +\\
&\qquad \omega_{12}^1 \wedge \xi^1 + \omega_{22}^1 \wedge \xi^2,\\
d\sigma_2^2 &= \phi_1^1 \wedge \sigma_2^2 - \psi_1^1 \wedge \sigma_2^2 - \psi_2^1 \wedge \theta^1 + \frac{1}{3}\lambda_{21}^1 \wedge \theta^2 +\\
&\qquad \omega_{12}^1 \wedge \xi^2 + \xi^1 \wedge \sigma_2^1.
\end{aligned} \tag{8}$$

The structure equations (8) do not contain any torsion coefficient depending on the group parameters. The first reduced character is $s_1' = 5$, and the degree of indeterminancy is 2. The Cartan involutivity test is not satisfied. Therefore we should use the procedure of prolongation, which gives us the following structure equations.

$$\begin{aligned}
d\theta^1 &= \eta^1 \wedge \theta^1 - \frac{1}{2}\eta^3 \wedge \theta^2 + \xi^1 \wedge \sigma_2^2 + \xi^2 \wedge \sigma_2^1,\\
d\theta^2 &= \eta^1 \wedge \theta^2 + \eta^2 \wedge \theta^2 - \theta^1 \wedge \xi^1 + \xi^2 \wedge \sigma_2^2,\\
d\xi^1 &= \eta^2 \wedge \xi^1 + \eta^3 \wedge \xi^2,\\
d\xi^2 &= 2\eta^2 \wedge \xi^2,\\
d\sigma_2^1 &= \eta^1 \wedge \sigma_2^1 - 2\eta^2 \wedge \sigma_2^1 - \frac{3}{2}\eta^3 \wedge \sigma_2^2 + \eta^4 \wedge \theta^1 +\\
&\qquad \eta^5 \wedge \xi^1 + \eta^6 \wedge \xi^2,\\
d\sigma_2^2 &= \eta^1 \wedge \sigma_2^2 - \eta^2 \wedge \sigma_2^2 - \eta^3 \wedge \theta^1 + \frac{1}{3}\eta^4 \wedge \theta^2 +\\
&\qquad \eta^5 \wedge \xi^2 + \xi^1 \wedge \sigma_2^1.\\
d\eta_1 &= \frac{1}{2}\eta^3 \wedge \xi^1 + \eta^4 \wedge \xi^2,\\
d\eta_2 &= \frac{2}{3}\eta^4 \wedge \xi^2,\\
d\eta_3 &= \frac{2}{3}\eta^4 \wedge \xi^1 - \eta^2 \wedge \eta^3,\\
d\eta_4 &= -2\eta^2 \wedge \eta^4,\\
d\eta_5 &= -\pi_1 \wedge \xi^2 - \eta^6 \wedge \xi^1 + 2\eta^3 \wedge \sigma_2^1 - 2\eta^4 \wedge \sigma_2^2 + \eta^1 \wedge \eta^5 - 3\eta^2 \wedge \eta^5,\\
d\eta_6 &= -\pi_1 \wedge \xi^1 - \pi_2 \wedge \xi^2 - \frac{10}{3}\eta^4 \wedge \sigma_2^1 + \eta^1 \wedge \eta^6 - 4\eta^2 \wedge \eta^6 - \frac{5}{2}\eta^3 \wedge \eta^5.
\end{aligned} \tag{9}$$

The forms $\eta_1, ..., \eta_6$ depend on differentials of the parameters of $\mathcal{H}$, while the forms π_1, π_2 depend on differentials of the prolongation variables.
In structure equations (9), the degree of indeterminancy is 2 and the reduced characters of the coframe are $s'_1 = 2, s'_2 = ... = s'_{12} = 0$. Since the Cartan involutivity test for the lifted coframe $\{\theta^1, \theta^2, \xi^1, \xi^2, \sigma^1_2, \sigma^2_2, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6\}$ is satisfied, then the coframe is involutive. Also all the essential torsion coefficients in the structure equations (9) are constants, then from the Theorem 11.8 of [5], we have:

Theorem 2.1. *The equation $u_t = \kappa u_x + u_{xx}$ is equivalent to the $u_t = u_{xx}$ under a contact transformation.*

Case-2:

Suppose $Q = \kappa u_2 + \lambda$, is a linear function ($\kappa \neq 0$).

In this case the analysis of the structure equations gives the following extra normalizations to (7).

$$\begin{aligned}
a^1_1 &= 4\kappa(4\kappa^2 u_1 u_2 + 4\kappa\lambda u_1 - 4\kappa p^2_2)^{-\frac{2}{3}},\\
b^1_1 &= \frac{1}{2}\kappa(4\kappa^2 u_1 u_2 + 4\kappa\lambda u_1 - 4\kappa p^2_2)^{\frac{1}{3}},\\
b^1_2 &= \frac{1}{2}(\kappa u_2 + \lambda)(4\kappa^2 u_1 u_2 + 4\kappa\lambda u_1 - 4\kappa p^2_2)^{\frac{1}{3}},\\
f^1_{21} &= -8\kappa u_1(4\kappa^2 u_1 u_2 + 4\kappa\lambda u_1 - 4\kappa p^2_2)^{-\frac{2}{3}},\\
g^1_{12} &= 6\sqrt[3]{2}\kappa u_1(\kappa^2 u_1 u_2 + \kappa\lambda u_1 - \kappa p^2_2)^{-\frac{2}{3}},
\end{aligned} \tag{10}$$

The expression for g^1_{22} are too long to be written out in full here. Now, all the group parameters are expressed as functions of the local coordinates $\{x_1, x_2, u_1, u_2, , p^1_2, p^2_2\}$. After normalization (10) the structure equations of coframe $\{\theta^1, \theta^2, \xi^1, \xi^2, \sigma^1_2, \sigma^2_2\}$, is

$$\begin{aligned}
d\theta^1 &= -\frac{2I}{3}\theta^1 \wedge \xi^1 - \frac{1}{3}\theta^1 \wedge \sigma^2_2 + \theta^2 \wedge \xi^2 + \xi^1 \wedge \sigma^2_2 + \xi^2 \wedge \sigma^1_2,\\
d\theta^2 &= -\theta^1 \wedge \xi^1 - \frac{I}{3}\theta^2 \wedge \xi^1 - \frac{1}{6}\theta^2 \wedge \sigma^2_2 + \xi^2 \wedge \sigma^2_2,\\
d\xi^1 &= \theta^2 \wedge \xi^2 + \frac{1}{6}\xi^1 \wedge \sigma^2_2,\\
d\xi^2 &= -\frac{2I}{3}\xi^1 \wedge \xi^2 + \frac{1}{3}\xi^2 \wedge \sigma^2_2,\\
d\sigma^1_2 &= \theta^1 \wedge \xi^1 - 8I\theta^1 \wedge \xi^2 - I\theta^2 \wedge \xi^2 - \frac{1}{2}\theta^2 \wedge \sigma^2_2 + 40I\xi^1 \wedge \xi^2 +\\
&\qquad \frac{4I}{3}\xi^1 \wedge \sigma^1_2 - 13\xi^2 \wedge \sigma^2_2 - \frac{2}{3}\sigma^1_2 \wedge \sigma^2_2,\\
d\sigma^2_2 &= 6\theta^1 \wedge \xi^2 - \theta^2 \wedge \xi^1 - 2\theta^2 \wedge \xi^2 + \xi^1 \wedge \sigma^1_2 + I\xi^1 \wedge \sigma^2_2,
\end{aligned} \tag{11}$$

where

$$I = \frac{\sqrt[3]{2}(\kappa^2 u^2 u_x + 2\kappa\lambda u u_x - \kappa u u_t - \kappa(u_x)^2 + \lambda^2 u_x - \lambda u_t + u_{tx})\kappa}{\sqrt[3]{(\kappa^2 u u_x + \kappa\lambda u_x - \kappa u_t)^4}}$$

is the only invariant of the symmetry group o equations of the from **Case-2**.
Note that, the exterior differential of I is

$$dI = \frac{1}{2}\theta^2 + \frac{4I^2}{3}\xi^1 + 6\xi^2 + \frac{1}{2}\sigma^1_2 + \frac{2I}{3}\sigma^2_2.$$

All derived invariants of the group are expressed as functions of I. Therefore, the rank of the coframe, is 1 and our manifold is 6-dimensional and by theorem 8.22 from [5], we deduce the following theorem.

Theorem 2.2. *The equation $u_t = (\kappa u + \lambda)u_x + u_{xx}$, ($\kappa \neq 0$) admits a contact transformation symmetry group of dimension 5.*

If Q is not a linear function, the analysis of the structure equations gives the following normalizations in addition to (7).

$$\begin{aligned} a_1^1 &= \frac{4\left(\frac{d^2Q}{du^2}\right)^2}{\left(\frac{dQ}{du}\right)^3}, \qquad b_1^1 = -\frac{1}{2}\frac{4\left(\frac{dQ}{du}\right)^2}{\left(\frac{d^2Q}{du^2}\right)}, \\ b_2^1 &= -\frac{1}{2}\frac{\left(\frac{dQ}{du}\right)\left(-2\left(\frac{d^2Q}{du^2}u_x\right)+\left(\frac{dQ}{du}\right)Q\right)}{\left(\frac{d^2Q}{du^2}\right)}. \end{aligned} \tag{12}$$

The expression for $f_{21}^1, g_{22}^1, g_{12}^1$ are too long to be written out in full here. Now, all the group parameters are expressed as functions of the local coordinate.
In case that, $\frac{d^3Q}{du^3}$ is nonzero, then an invariant of the form:

$$2(\frac{d^2Q}{du^2})^2 - (\frac{d^3Q}{du^3})(\frac{dQ}{du}) \tag{13}$$

could be computed at which vanishing or non-vanishing of this invariant leads into different results. In what follows, we will investigate these issues. Note that, according to our computations, vanishing of (13) or being $Q(u)$ of quadratic type leads, on the type and number of these invariants, into the same results.

Case-3:
$Q(u)$ is a quadratic polynomial or

$$2(\frac{d^2Q}{du^2})^2 - (\frac{d^3Q}{du^3})(\frac{dQ}{du}) = 0, \qquad \frac{d^3Q}{du^3} \neq 0. \tag{14}$$

If (14) satisfied then the structure equations of the coframe are

$$\begin{aligned} d\theta^1 &= J_3\theta^1\wedge\xi^2 - 8J_1\theta^2\wedge\xi^1 + J_2\theta^2\wedge\xi^2 + \xi^1\wedge\sigma_2^2 + \xi^2\wedge\sigma_2^1, \\ d\theta^2 &= -\theta^1\wedge\xi^1 + \frac{J_3}{3}\theta^2\wedge\xi^2 + \xi^2\wedge\sigma_2^2, \\ d\xi^1 &= \theta^1\wedge\xi^2 + \theta^2\wedge\xi^2 - (16J_1 + \frac{J_3}{2})\xi^1\wedge\xi^2, \\ d\xi^2 &= 0, \\ d\sigma_2^2 &= -16J_1\theta^1\wedge\xi^1 + 2(20J_1 - J_2)\theta^1\wedge\xi^2 - J_2\theta^2\wedge\xi^1 - \theta^2\wedge\sigma_2^2 - \\ &\quad (64{J_1}^2 + 4J_1J_3 - 8J_1 + J_2)\theta^2\wedge\xi^2 + \xi^1\wedge\sigma_2^1 - \xi^2\wedge\sigma_2^1 - \\ &\quad \frac{J_3}{2}\xi^2\wedge\sigma_2^2. \end{aligned} \tag{15}$$

The expression for $d\sigma_2^2$ is too long to be written out in full here.
If $Q(u)$ is a quadratic polynomial then the structure equations of coframe is different

from (15), and expressed only by $\{J_1, J_2, J_3\}$, where

$$
\begin{aligned}
J_1 &= \frac{\left(\frac{d^2Q}{du^2}\right)^3 \left(\left(\frac{d^2Q}{du^2}\right) {u_x}^2 - Q\left(\frac{dQ}{du}\right) u_x + \left(\frac{dQ}{du}\right) u_t\right)}{\left(\frac{dQ}{du}\right)^6}, \\
J_2 &= -\frac{4\left(\frac{d^2Q}{du^2}\right)^3}{\left(\frac{dQ}{du}\right)^9}\Bigg(\left(\frac{dQ}{du}\right)^4 Q u_x + \left(\frac{d^2Q}{du^2}\right)\left(\frac{dQ}{du}\right)^3 {u_x}^2 - \\
&\quad 2\left(\frac{d^2Q}{du^2}\right)\left(\frac{dQ}{du}\right)^2 Q^2 u_x + 6\left(\frac{d^2Q}{du^2}\right)^2\left(\frac{dQ}{du}\right) Q {u_x}^2 - \\
&\quad 4\left(\frac{d^2Q}{du^2}\right)^3 {u_x}^3 + 2\left(\frac{d^2Q}{du^2}\right)\left(\frac{dQ}{du}\right)^2 Q u_t - 6\left(\frac{d^2Q}{du^2}\right)^2\left(\frac{dQ}{du}\right) u_x u_t \\
&\quad -2\left(\frac{d^2Q}{du^2}\right)\left(\frac{dQ}{du}\right)^2 u_{tx} - \left(\frac{dQ}{du}\right)^4 u_t\Bigg), \qquad (16) \\
J_3 &= -\frac{8\left(\frac{d^2Q}{du^2}\right)^2\left(\left(\frac{dQ}{du}\right)^3 u_x + 2\left(\frac{d^2Q}{du^2}\right)^3 {u_x}^2 - 2\left(\frac{d^2Q}{du^2}\right)\left(\frac{dQ}{du}\right) Q u_x + 2\left(\frac{d^2Q}{du^2}\right)\left(\frac{dQ}{du}\right) u_t\right)}{\left(\frac{dQ}{du}\right)^6},
\end{aligned}
$$

are invariants of the symmetry group of an equation from **Case-3**.
All derived invariants of the group are expressed as functions of $\{J_1, J_2, J_3\}$. Therefore the rank of the coframe, is 3. Again by theorem 8.22 from [5], we have

Theorem 2.3. *If $Q(u)$ is a quadratic polynomial or satisfy $2(\frac{d^2Q}{du^2})^2 - (\frac{d^3Q}{du^3})(\frac{dQ}{du}) = 0$, $(\frac{d^3Q}{{du_x}^3} \neq 0)$ then, the equation $u_t = Q(u)u_x + u_{xx}$, admits a contact transformation symmetry group of dimension 3.*

Case-4:
We will make the following assumption for (1):

$$
2(\frac{d^2Q}{du^2})^2 - (\frac{d^3Q}{du^3})(\frac{dQ}{du}) \neq 0, \qquad \frac{d^3Q}{du^3} \neq 0. \tag{17}
$$

The structure equations of the coframe, in this case, is

$$
\begin{aligned}
d\theta^1 &= -J_4\theta^1\wedge\theta^2 + (4J_1J_3J_4 + J_3 + 32{J_1}^2J_4 + \frac{1}{8}{J_3}^2J_4 + 8J_1J_4)\theta^1\wedge\xi^2 - \\
&\quad (16{J_1}^2J_4 + 2J_1J_3J_4 + \frac{1}{16}{J_3}^2J_4 - 8J_1)\theta^2\wedge\xi^1 + \frac{J_4(16J_1+J_3)}{2}\theta^1\wedge\xi^1 + \\
&\quad (64{J_1}^3J_4 + \frac{1}{64}{J_3}^3J_4 + 16{J_1}^2J_4 + 12{J_1}^2J_3J_4 + \frac{3}{4}J_1{J_3}^2J_4 + J_1J_3J_4 + \\
&\quad J_2)\theta^2\wedge\xi^2 + \xi^1\wedge\sigma_2^2 + \xi^2\wedge\sigma_2^1, \\
d\theta^2 &= -\theta^1\wedge\xi^1 + \frac{J_4(16J_1+J_3)}{4}\theta^2\wedge\xi^1 + (2J_1J_3J_4 + \frac{1}{2}J_3 + 16{J_1}^2J_4 + \\
&\quad \frac{1}{16}{J_3}^2J_4 + 4J_1J_4)\theta^2\wedge\xi^2 + \xi^2\wedge\sigma_2^2, \qquad (18) \\
d\xi^1 &= \theta^1\wedge\xi^2 - \frac{J_4}{2}\theta^2\wedge\xi^1 + (1 - 4J_1J_4 - \frac{1}{4}J_3J_4)\theta^2\wedge\xi^2 + \\
&\quad (2J_1J_3J_4 - 16J_1 - \frac{1}{2}J_3 + 16{J_1}^2J_4 + \frac{1}{16}{J_3}^2J_4 - 4J_1J_4))\xi^1\wedge\xi^2, \\
d\xi^2 &= -J_4\theta^2\wedge\xi^2 + \frac{J_4(16J_1+J_3)}{2}\xi^1\wedge\xi^2.
\end{aligned}
$$

The expression for $d\sigma_2^1, d\sigma_2^2$ are too long to be written out in full here.
There is an invariant extra to (16), for the symmetry group of equations from **case4**, which is

$$J_4 = \frac{2(\frac{d^2Q}{du^2})^2 - (\frac{d^3Q}{du^3})(\frac{dQ}{du})}{(\frac{d^2Q}{du^2})^2}.$$

All derived invariants of the group are functionally expressed as functions of $\{J_1, J_2, J_3, J_4\}$. The rank of the coframe, is 4, therefore we have:

Theorem 2.4. *If $Q(u)$ satisfy, $2(\frac{d^2Q}{du^2})^2 - (\frac{d^3Q}{du^3})(\frac{dQ}{du}) \neq 0$ and $(\frac{d^3Q}{du_1^3} \neq 0)$ then, the equation $u_t = Q(u)u_x + u_{xx}$, admits a contact transformation symmetry group of dimension 2.*

3. Conclusion

In this paper, the moving coframe method of [4] is applied to the local equivalence problem for a class of systems of the general form of Burgers' equations under the action of a pseudo-group of contact transformations. We have found four subclasses and showed that every type of the general form of Burgers' equations belongs to a system from one of these subclasses. The equivalence condition of first subclass, structure equations and invariants for all subclasses are found.

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