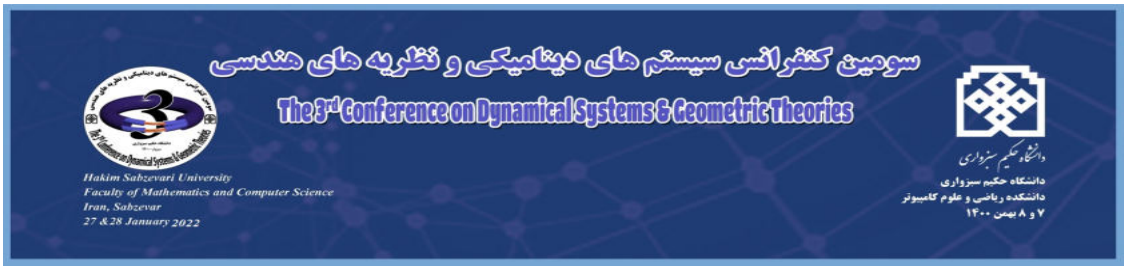

Chaotic Dynamics of Lorenz-type systems

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Abstract. In the present paper, we consider a family of one-dimensional discontinuous monotone dynamical systems of the interval $[0, 1]$ onto itself, with $N = 2$ branches and investigate their chaotic dynamics. Here we follow the definition suggested by Devaney in order to show robust full chaos.

Keywords: chaos, piecewise smooth dynamical systems, expanding

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1. Introduction

The study of piecewise smooth systems had a wide expansion in the last decade. This is due to the large number of physical and engineering systems with nonsmooth vector fields. Many applications come from power electronic circuits in electrical engineering, which gave a wide impulse to the study of piecewise defined systems, both continuous and discontinuous. Several kinds of bifurcations of non-smooth systems also appear in forced impact oscillators, in mechanical engineering, in economics and social sciences.

Piecewise smooth (PWS for short) dynamical systems are characterized by the fact that their state space is divided into partitions by borders also denoted as switching manifolds. Within each partition, the system is smooth (that is C^k up to some k) but the rules which govern the dynamic behavior (that is the right hand side of the system function) change at the boundaries ([1]).

In the present work, we consider a map of an interval I into itself. In the last decades, the condition of persistence of chaos in the whole interval (robust full chaos) has become very important in engineering applications, especially those related to grazing bifurcations, for security transmissions, as well as in other applied fields. In such applications, the systems are often ultimately described by piecewise smooth maps. In particular, it is known that the three-dimensional ordinary differential equations called Lorenz flows and discontinuity-induced bifurcation can be analyzed by using suitable Poincaré maps, which are often piecewise smooth and discontinuous. An important class of such systems, for

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which the Poincaré sections are maps with two branches ($N = 2$), leads to a family of discontinuous maps of an interval, with two increasing branches, called Lorenz maps of Class $\mathcal{A}$ and, as we shall recall, already considered by many authors.

Moreover, particularly important is to investigate the conditions of full chaos, and its robustness, in these kind of maps (i.e. discontinuous and with increasing branches). For the class of expanding Lorenz maps this has been considered in the literature. In fact, a well known sufficient, but not necessary, condition of robust full chaos for an expanding Lorenz map $f(x)$ is $f'(x) > \sqrt{2}$ for any x (seen in [6], [5]), while the necessary and sufficient conditions for the case $1 < f'(x) < \sqrt{2}$ can be considered outlined in [2].

Differently, the case of a piecewise smooth map with $N > 2$ branches, has got less attention up to now. Besides the basic results related to the piecewise linear map with constant slope, expanding piecewise monotone maps with $N > 2$ branches have been considered by Li and Yorke in [4], where some relevant properties of the chaotic sets are determined, but not the characterization of full chaos.

Still less attention has been paid to the robustness of the chaotic regime. In many applications it is relevant to get robust chaos, i.e. structurally stable chaos, or persistent under parameter perturbations. In particular, the occurrence and robustness of full chaos in a Poincaré map which is a *non-expanding Lorenz map*, can be considered as an open problem. Indeed this is a relevant case, also in applications, which motivates the present work. Besides the class of Lorenz maps, we are interested in a particular class of piecewise monotone discontinuous maps with $N > 2$ branches, which is associated with the first return map in Lorenz maps. Their peculiarity is that the internal branches of the first return maps are *onto* the interval, and we call these maps Baker-like. A relevant fact is that even if a Lorenz map is not expanding, its related first return map may be an expanding Lorenz map or Baker-like map, and this allows to get results otherwise difficult to prove.

So, the main objective of this work is to give the necessary and sufficient conditions for a discontinuous piecewise smooth *expanding* map f of an interval into itself, constituted by N pieces with $N = 2$, to be robustly chaotic in the whole interval. As recalled above, for $N = 2$ the map is a Lorenz map of Class $\mathcal{A}$, and this problem has been investigated by other authors, mainly giving sufficient conditions for full chaos.

2. Main results

First of all, we state some basic notations related to the chaos theory. Many definitions of chaos can be found in the existing literature. We will follow here the definition that was suggested by Devaney in 1986. It has three ingredients defined as follows:

Definition 2.1. (Chaos) *Let (X, d) be a metric space without isolated points. Then a dynamical system $f : X \longrightarrow X$ is said to be chaotic (in the sense of Devaney) if it satisfies the following conditions:*

(1) **transitivity:** *f is topologically transitive in X; that is, for any pair of non-empty open sets U and V of X there exists a natural number n such that $f^n(U) \cap V \neq \varnothing$;*

(2) **density:** *the periodic points of f are dense in X;*

(3) **sensitivity:** *f has sensitive dependence on initial conditions in X; that is, there is a positive constant δ (sensitivity constant) such that for every point x of X and every neighborhood N of x there exists a point y in N and a non negative integer n such that $d(f^n(x), f^n(y)) \geq \delta$.*

For convenience, suppose that I is the interval $[0,\ 1]$.

Definition 2.2. (Robust or Structurally stable choas) *A map* $\phi(x;p) := I \longrightarrow I$ *depending on a vector of parameters* $p \in P$ *is said to be robustly chaotic (or structurally stable chaotic) in* X *at* p_0 *if the same property holds in a neighborhood* $U(p_0) \cap P$ *of* p_0 *(i.e. the maps are topologically conjugate).*

In this work, we consider a one dimensional discontinuous piecewise monotone $\mathcal{C}^{(1)}$ map $f : I \longrightarrow I$ of the interval $[0, 1]$ onto itself, with $N \geq 2$ branches. That is, Lorenz maps for $N = 2$ and a family of expanding Baker-like maps for $N > 2$. We distinguish between maps with one discontinuity point, i.e. the number of branches is $N = 2$, which lead to the well known class of Lorenz maps, or more branches, $N \geq 3$, which we call Baker-like maps. This class of maps is relevant because it may represent the first return map of non-expanding Lorenz maps. We determine the necessary and sufficient conditions to have robust chaos, with robust we mean persistence under parameter perturbations and full chaos refers to the whole interval.

Definition 2.3. (Lorenz map) *A Lorenz map* $x \longmapsto f(x)$ *is defined by a function* $f : I \longrightarrow I$, $I = [0, 1]$, *with a single discontinuity point* $\xi_1 \in (0, 1)$:

$$f(x) = \begin{cases} f_1(x) & \text{if } 0 \leq x < \xi_1 \\ f_2(x) & \text{if } \xi_1 \leq x \leq 1 \end{cases} \tag{1}$$

such that $f_i(x)$ *are strictly increasing* $\mathcal{C}^{(1)}$ *functions in* I_i, $i = 1, 2$, *with* $f_1(\xi_1) = 1$ *and* $f_2(\xi_1) = 0$, $I_1 = [0, \xi_1]$, $I_2 = [\xi_1, 1]$.

Definition 2.4. (expanding) *A piecewise* $\mathcal{C}^{(1)}$ *Lorenz map (1) is called expanding if a constant* $\lambda > 1$ *exists such that* $f_i'(x) > \lambda$ *for any* $x \in I_i$ and $1 \leq i \leq N$.

Notice that in our definition of a Lorenz map, it is $f([0, 1]) = [0, 1)$, but as it is common when dealing with discontinuous maps, we can say the map is chaotic in closed interval $[0, 1]$.

In the following, writing a condition on the derivative as $f'(x)$, for any $x \in I$, we mean that for the components f_i the property holds in the closed interval of definiton.

As we have already mentioned in the introduction, we know, from [6], that for an expanding Lorenz map the condition $f'(x) > \sqrt{2}$ is sufficient but not necessary to have chaos in $[0, 1]$. In this section we show that the necessary and sufficient conditions, to have full chaos, for $1 < \lambda < \sqrt{2}$ are related to the existing basic cycle RL^{n-1} or $R^{n-1}L$, for some $n > 1$, which must be homoclinic. We emphasize that the existence of such basic cycles is related to a border collision bifurcation, clearly distinguished from the homoclinic bifurcation values.

Proposition 2.5. *([3]). An expanding Lorenz map* f *is chaotic in* $[0, 1]$ *iff the existing basic cycle (with symbolic sequence* $L^{n-1}R$ *or* $R^{n-1}L$, *for* $n > 1$*) is homoclinic. Chaos is not robust at the homoclinic bifurcation values detected via the conditions* $f^n(0) = x^n_{\max}$ *and* $f^n(1) = x^n_{\min}$, *where* $x^n_{\min}$ *and* $x^n_{\max}$ *are the minimum and the maximum of the periodic points of the* $n-$*cycle.*

The homoclinic bifurcations related to the basic cycles having symbolic sequence RL^{n-1} and $R^{n-1}L$, $n > 1$, have been considered also in [2] for the $\beta-$Transformation $T(x)$. Moreover, similar results hold for a generic expanding Lorenz map (since it is topologically conjugate to a $\beta-$Transformation $T(x)$.

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