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The entropy of relative dynamical systems having countably many atoms

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Abstract. In this paper, in order to develop a mathematical model underlying uncertainty and fuzziness in a dynamical system, which is called relative mathematical modeling, we are going to apply the notion of observer. First, by using a mathematical model of a one dimensional observer, the notion of relative entropy for a relative dynamical system having countably many atoms is considered. Also, some ergodic properties of relative dynamical systems are investigated. At the end, a new version of Kolmogorov-Sinai theorem for a relative dynamical system having countably many atoms is given.

Keywords: Entropy, observer, relative mathematical modeling, relative dynamical system

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1. Introduction

Entropy is applicable and useful in studying the behavior of stochastic processes since it represents the ambiguity and disorder of the processes without being restricted to the forms of the theoretical probability distributions. Different entropy measures have been studied and presented including Shannon entropy, Renyi entropy, Tsallis entropy, Sample entropy, Permutation entropy, Approximate entropy, and Transfer entropy. Since in mathematical modeling of physical systems the role of observer is important, so a method is needed to measure the entropy of a system from the point of view of an observer. Any mathematical model according to the view point of an observer is called a relative model [?, ?]. The notion of a relative dynamical system as a generalization of a fuzzy dynamical system has been defined in [?]. Also, the concept of entropy of a relative dynamical system has been introduced in [?, ?]. This article is an attempt to present a new approach to the entropy of relative dynamical systems having countably many atoms.

2. Basic Notions

This section is devoted to provide some basic notions of relative structures. A modeling for an observer of a set X is a fuzzy set $\Theta : X \to [0, 1]$ [?]. In fact this kinds of fuzzy

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sets are called " one dimentional observes". The idea is based on the relation between "experiance " and "information" from the view point of an observer. Let Θ be an observer on X, then we say $\lambda \subseteq \Theta$ if $\lambda(x) \leq \Theta(x)$ for all $x \in X$. Moreover, if $\lambda_1, \lambda_2 \subseteq \Theta$ then $\lambda_1 \vee \lambda_2$ and $\lambda_1 \wedge \lambda_2$ are subsets of Θ, and defined by

$$(\lambda_1 \vee \lambda_2)(x) = \sup\{\lambda_1(x), \lambda_2(x)\},$$

and

$$(\lambda_1 \wedge \lambda_2)(x) = \inf\{\lambda_1(x), \lambda_2(x)\},$$

where $x \in X$.

Definition 2.1. A collection F_Θ of subsets of Θ is said to be a σ_Θ-algebra in Θ if F_Θ satisfies the following conditions [?],

(i) $\Theta \in F_\Theta$,
(ii) $\lambda \in F_\Theta$ then $\lambda^{'} = \Theta - \lambda \in F_\Theta$. $\lambda^{'}$ is the complement of λ with respect to Θ,
(iii) if $\{\lambda_i\}_{i=1}^{\infty}$ is a sequence in F_Θ then $\vee_{i=1}^{\infty}\lambda_i = \sup_i \lambda_i \in F_\Theta$,
(iv) $\dfrac{\Theta}{2}$ doesn't belong to F_Θ.

If P_1 and P_2 are σ_Θ-algebras on X then $P_1 \vee P_2$ is the smallest σ_Θ-algebra that contains $P_1 \cup P_2$, denoted by $[P_1 \cup P_2]$.

Definition 2.2. A positive $\Theta-$measure m_Θ over F_Θ is a function $m_\Theta : F_\Theta \to I$ which is countably additive. This means that if λ_i is a disjoint countable collection of members of F_Θ, (i.e. $\lambda_i \subseteq \lambda_j^{'} = \Theta - \lambda_j$ whenever $i \neq j$) then

$$m_\Theta(\vee_{i=1}^{\infty}\lambda_i) = \sum_{i=1}^{\infty} m_\Theta(\lambda_i).$$

The $\Theta-$measure m_Θ has the following properties [?],

(i) $m_\Theta(\chi_\emptyset) = 0$,
(ii) $m_\Theta(\lambda^{'} \vee \lambda) = m_\Theta(\Theta)$ and $m_\Theta(\lambda^{'}) = m_\Theta(\Theta) - m_\Theta(\lambda)$ for all $\lambda \in F_\Theta$,
(iii) $m_\Theta(\lambda \vee \mu) + m_\Theta(\lambda \wedge \mu) = m_\Theta(\lambda) + m_\Theta(\mu)$ for each $\lambda, \mu \in F_\Theta$,
(iv) m_Θ is a nondecreasing function i.e. if $\lambda, \eta \in F_\Theta$ and $\lambda \subseteq \Theta$, then $m_\Theta(\lambda) \leq m_\Theta(\eta)$.

The triple (X, F_Θ, m_Θ) is called a $\Theta-$ measure space and the elements of F_Θ are called relative measurable sets. The $\Theta-$ measure space, (X, F_Θ, m_Θ), is called a relative probability $\Theta-$measure space if $m_\Theta(\Theta) = 1$ [?].

Example 2.3. Let (X, β, p) be a classical probability measure space and $\Theta = \chi_X$. Then $F_\Theta = \{\chi_A : A \in \beta\}$ is a σ_Θ-algebra on X. Define $m_\Theta(\chi_A) = p(A)$, $A \in \beta$. Then (X, F_Θ, m_Θ) is a relative probability $\Theta-$ measure space.

Definition 2.4. Let (X, F_Θ, m) be a $\Theta-$measure space, the elements μ,λ of F_Θ are called m_Θ-disjoint if $m_\Theta(\lambda \wedge \mu) = 0$.

A $\Theta-$relation '=(mod m_Θ)' on F_Θ is defined as bellow

$$\lambda = \mu (\mathrm{mod}\, m_\Theta) \quad \text{iff} \quad m_\Theta(\lambda) = m_\Theta(\mu) = m_\Theta(\lambda \wedge \mu),$$

for each $\lambda, \mu \in F_\Theta$.
$\Theta-$relation '=(mod m_Θ)' is an equivalence relation. $\tilde{F_\Theta}$ denotes the set of all equivalence classes induced by this relation, and $\tilde{\mu}$ is the equivalence class determined by μ. For $\lambda, \mu \in F_\Theta$, $\lambda \wedge \mu = 0$ (mod m_Θ) iff λ, μ are m_Θ-disjoint. We shall identify $\tilde{\mu}$ with μ.

Definition 2.5. Let (X, F_Θ, m_Θ) be a $\Theta-$measure space, and P be a sub-σ_Θ-algebra of F_Θ. Then an element $\tilde{\lambda} \in \tilde{P}$ is an atom of P if

(i) $m_\Theta(\lambda) > 0$,
(ii) for each $\tilde{\mu} \in \tilde{P}$such that $m_\Theta(\lambda \wedge \mu) = m_\Theta(\mu) \neq m_\Theta(\lambda)$ then $m_\Theta(\mu) = 0$.

Theorem 2.6. *Let (X, F_Θ, m_Θ) be a $\Theta-$measure space, and P be a sub-σ_Θ-algebra of F_Θ. If $\tilde{\lambda_1}$,$\tilde{\lambda_2}$ are disjoint atoms of P then they are m_Θ-disjoint.*

3. Entropy of a sub-σ_Θ-algebra with countable atoms

In this section we introduce the notion of entropy of a sub-σ_Θ-algebra with countable atoms. At the following, the set of all sub-σ_Θ-algebra of F_Θ with countable atoms is denoted by $R^*(F_\Theta)$. Assume that F_Θ is a σ_Θ-algebra and $P_1, P_2 \in R^*(F_\Theta)$, and $\{\lambda_i; i \in \mathbb{N}\}$ and $\{\mu_j; j \in \mathbb{N}\}$ denote the atoms of P_1 and P_2 respectively, then the atoms of $P_1 \vee P_2$ are $\lambda_i \wedge \mu_j$ which $m_\Theta(\lambda_i \wedge \mu_j) > 0$ for each $i, j \in \mathbb{N}$.
If $\gamma \in \bar{F_\Theta}$ we set

$$P_1 \vee \gamma = \{\lambda_i \wedge \gamma; m_\Theta(\lambda_i \wedge \gamma) > 0, i \in \mathbb{N}\}.$$

Theorem 3.1. *Let $\{\lambda_i; i \in \mathbb{N}\}$ be a m_Θ-disjoint collection of relative measurable sets of relative probability Θ-measure space (X, F_Θ, m_Θ), then,*

$$m_\Theta(\vee_{i=1}^{\infty}(\lambda_i)) = \sum_{i=1}^{\infty} m_\Theta(\lambda_i).$$

Definition 3.2. Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space and $P_1, P_2 \in R^*(F_\Theta)$. We say that P_2 is an m_Θ-refinement of P_1, denoted by $P_1 \leq_{m_\Theta} P_2$, if for each $\mu \in \bar{P_2}$ there exists $\lambda \in \bar{P_1}$ such that,

$$m_\Theta(\lambda \wedge \mu) = m_\Theta(\mu).$$

Theorem 3.3. *Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space and $P_1, P_2, P_3 \in R^*(F_\Theta)$ if $P_1 \leq_{m_\Theta} P_2$ then,*

$$P_1 \vee P_3 \leq_{m_\Theta} P_2 \vee P_3.$$

Definition 3.4. Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space, and P be a sub σ_Θ-algebra of F_Θ which $P \in R^*(F_\Theta)$, the entropy of P is defined as

$$H_\Theta(P) = -\log \sup_{i \in \mathbb{N}} m_\Theta(\mu_i),$$

where $\{\mu_i; i \in \mathbb{N}\}$ are atoms of P.

Definition 3.5. Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space and $P \in R^*(F_\Theta)$. The conditional entropy of P given $\gamma \in \bar{F_\Theta}$ is defined by

$$H_\Theta(P|\gamma) = -\log \sup_{i \in \mathbb{N}} m_\Theta(\mu_i|\gamma),$$

where,

$$m_\Theta(\mu_i|\gamma) = \frac{m_\Theta(\mu_i \wedge \gamma)}{m_\Theta(\gamma)} \qquad (m_\Theta(\gamma) \neq 0).$$

Theorem 3.6. *Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space, and $P_1, P_2 \in R^*(F_\Theta)$ which $\bar{P_1} = \{\lambda_i; i \in \mathbb{N}\}$ and $\bar{P_2} = \{\mu_j; j \in \mathbb{N}\}$. Then,*

(i) $P_1 \leq_{m_\Theta} P_2 \Rightarrow H_\Theta(P_1) \leq H_\Theta(P_2)$,
(ii) $P_1 \leq_{m_\Theta} P_2 \Rightarrow H_\Theta(P_1|\gamma) \leq H_\Theta(P_2|\gamma)$.

Definition 3.7. Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space and $P_1, P_2 \in R^*(F_\Theta)$. We say that P_1 and P_2 are m_Θ-equivalent, denoted by $P_1 \approx_{m_\Theta} P_2$, if the following axioms are satisfied:

(i) If $\lambda \in \bar{P_1}$ then $m_\Theta(\lambda \wedge (\vee\{\mu; \mu \in \bar{P_2}\})) = m_\Theta(\lambda)$.
(ii) If $\mu \in \bar{P_2}$ then $m_\Theta(\mu \wedge (\vee\{\lambda; \lambda \in \bar{P_1}\})) = m_\Theta(\mu)$.

Theorem 3.8. *Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space, and $P_1, P_2 \in R^*(F_\Theta)$. If $P_1 \approx_{m_\Theta} P_2$ then,*

$$P_1 \approx_{m_\Theta} P_1 \vee P_2.$$

Theorem 3.9. *Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space, and $P_1, P_2 \in R^*(F_\Theta)$. If $P_1 \approx_{m_\Theta} P_2$ then,*

$$H_\Theta(P_1) \leq H_\Theta(P_1 \vee P_2).$$

Definition 3.10. Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space and $P \in R^*(F_\Theta)$. The diameter of P is defined as follows

$$\mathrm{diam}P = \sup_{\lambda_i \in \bar{P}} m_\Theta(\lambda_i).$$

Definition 3.11. Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space,and $P_1, P_2 \in R^*(F_\Theta)$, which $\bar{P_1} = \{\lambda_i; i \in \mathbb{N}\}$, $\bar{P_2} = \{\gamma_k; k \in \mathbb{N}\}$. The conditional entropy of P_1 given P_2 is defined as

$$\begin{aligned} H_\Theta(P_1|P_2) &= -\log \sup_{i \in \mathbb{N}} \frac{\mathrm{diam}(\lambda_i \vee P_2)}{\mathrm{diam}P_2} \\ &= -\log \sup_{j \in \mathbb{N}} \frac{\mathrm{diam}(P_1 \vee \mu_j)}{\mathrm{diam}P_2}. \end{aligned}$$

Theorem 3.12. *Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space, and $P_1, P_2, P_3 \in R^*(F_\Theta)$. Then,*

(i) $P_2 \leq_{m_\Theta} P_3 \Rightarrow H_\Theta(P_1|P_2) \leq H_\Theta(P_1 \vee P_3)$,
(ii) $H_\Theta(P_1|P_2) \leq H_\Theta(P_1 \vee P_2)$.

Theorem 3.13. *Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space, and $P_1, P_2, P_3 \in R^*(F_\Theta)$. If $P_1 \leq_{m_\Theta} P_2$ then,*

$$H_\Theta(P_1|P_3) \leq H_\Theta(P_2|P_3).$$

4. Entropy of a relative dynamical system having countably many atoms

Definition 4.1. Suppose (X, F_Θ, m_Θ) be a $\Theta-$measure space and Θ be a constant observer on X. A transformation $\varphi : (X, F_\Theta, m_\Theta) \to (X, F_\Theta, n_\Theta)$, is said to be a $\Theta-$measure preserving if $m_\Theta(\varphi^{-1}(\mu)) = n_\Theta(\mu)$ for all $\mu \in \bar{F_\Theta}$.

Theorem 4.2. *Suppose $\varphi : (X, F_\Theta, m_\Theta) \to (X, F_\Theta, n_\Theta)$, be a $\Theta-$measure preserving transformation. Then for each $P \in R^*(F_\Theta)$ we have,*

$$H_\Theta(P) = H_\Theta(\varphi^{-1}(P)).$$

Definition 4.3. Suppose $\varphi : (X, F_\Theta, m_\Theta) \to (X, F_\Theta, n_\Theta)$, be a $\Theta-$measure preserving transformation. If $P \in R^*(F_\Theta)$, we define the entropy of φ with respect to P as:

$$h_\Theta(\varphi, P) = \lim_{n \to \infty} \frac{1}{n} H_\Theta(\vee_{i=0}^{n-1} \varphi^{-i}(P)).$$

THEOREM 4.4. *Let $\varphi : (X, F_\Theta, m_\Theta) \to (X, F_\Theta, n_\Theta)$, be a $\Theta-$measure preserving transformation and $P \in R^*(F_\Theta)$. Then,*

(i) $h_\Theta(\varphi, \varphi^{-1}(P)) = h_\Theta(\varphi, P)$,

(ii) $h_\Theta(\varphi, \vee_{i=0}^{r-1} \varphi^{-i}(P)) = h_\Theta(\varphi, P)$ *for every* $r \geq 1$.

THEOREM 4.5. *Let $\varphi : (X, F_\Theta, m_\Theta) \to (X, F_\Theta, n_\Theta)$, be a $\Theta-$measure preserving transformation and $P_1, P_2 \in R^*(F_\Theta)$. Then,*

(i) $P_1 \leq_{m_\Theta} P_2 \Rightarrow h_\Theta(\varphi, P_1) \leq h_\Theta(\varphi, P_2)$,

(ii) *if $P_1, P_2 \in R^*(F_\Theta)$ such that $P_1 \approx_{m_\Theta} P_2$ then,*

$$\varphi^{-1}(P_1) \approx_{m_\Theta} \varphi^{-1}(P_2).$$

DEFINITION 4.6. The entropy of the relative dynamical system $(X, F_\Theta, m_\Theta, \varphi)$ is the number $h_\Theta(\varphi)$ defined by,

$$h_\Theta(\varphi) = \sup_P h_\Theta(\varphi, P),$$

where the supremum is taken over all sub-σ_Θ-algebras of F_Θ which $P \in R^*(F_\Theta)$.

DEFINITION 4.7. $P \in R^*(F_\Theta)$ is said to be a m_Θ-generator of the relative dynamical system $(X, F_\Theta, m_\Theta, \varphi)$ if there exists an integer $r > 0$ such that,

$$Q \leq_{m_\Theta} \vee_{i=0}^{r} \varphi^{-i} P,$$

for each $Q \in R^*(F_\Theta)$.

THEOREM 4.8. *If P is a m_Θ-generator of the relative dynamical system $(X, F_\Theta, m_\Theta, \varphi)$ then,*

$$h_\Theta(\varphi, Q) \leq h_\Theta(\varphi, P),$$

for each $Q \in R^*(F_\Theta)$.

Now we can deduce the following version of Kolmogorov-Sinai theorem for relative dynamical systems having countably many atoms.

THEOREM 4.9. *If P is a m_Θ-generator of relative dynamical system $(X, F_\Theta, m_\Theta, \varphi)$ then,*

$$h_\Theta(\varphi) = h_\Theta(\varphi, P).$$

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