Topological Asymptotic average Shadowing Property

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Abstract. We introduce topological definition of asymptotic average shadowing property. We show that this property implies that every two disjoint points in the space are proximal.

1. Introduction

The pseudo-orbit tracing property is one of the most important notions in dynamical systems, which is closely related to stability and chaos of systems . This concept is motivated by computer simulations. More precisely, let X be a set and $f : X \to X$ be a map. Then in the computation of f with initial value $x_0 \in X$, computer approximates $f(x_0)$ by some point x_1. To continue the process, it computes the value x_2 as an approximation of $f(x_1)$ and so on. For formulating this concept we have to use the 'distance' between points to control approximation errors. In a metric space (X, d) one can approximates points using metric d and define a pseudo-orbit with error δ as a sequence $x_0, x_1, \ldots$ with $d(x_{j+1}, f(x_j)) < \delta$ for all $j \geq 0$. However, for general topological spaces such a distance cannot be found unless we have somewhat more structure than what the topology itself provides. This issue will be solved if we consider a completely regular topological spaces which is equipped with an structure, called uniformity, enabling us to control the distance between points in these spaces. Using this structure, Das et al. generalized the usual definitions of shadowing, and chain recurrence for homeomorphisms to topological spaces. Then, we [1] proved that a dynamical system with ergodic shadowing is topologically chain transitive. Wu [2] introduced the topological concepts of weak uniformity, uniform rigidity, and multi-sensitivity and obtained some equivalent characterizations of uniform rigidity. Then, we [3] proved that a point transitive dynamical system in a Hausdorff uniform space is either almost (Banach) mean equicontinuous or (Banach) mean sensitive. Recently, we [?] generalized concepts of entropy points, expansivity and shadowing property for dynamical systems to uniform spaces and obtained a relation between topological shadowing property and positive uniform entropy. Good and Macías [4] obtained some equivalent characterizations and iteration invariance of various definitions of shadowing in the compact uniform spaces.

Nevertheless when calculating approximate trajectories, it makes sense to consider errors small on average, since controlling them in each iteration may be impossible. The notion of average pseudo-orbit introduced by Blank in 1988. In a metric space (X,d) an average pseudo-orbit with error δ is a sequence $x_0, x_1, \dots$ for which there is $N \in \mathbb{N}$ such that $\frac{1}{n}\sum_{j=0}^{n-1} d(x_{j+k+1}, f(x_{j+k})) < \delta$ for any $n \geq N$ and $k \in \mathbb{N}$. The average shadowing property is related to finding an averagely close real orbit for any average pseudo-orbit. But in a general topological space we need some method to control the average of errors in a pseudo-orbit . Motivated by mentioned ideas, We introduced average shadowing property on uniform spaces. [5]

Here we show that asymptotic average shadowing property can be defined in a natural way on uniform spaces. In order to do this, we control the average of errors of a pseudo-orbit in a non-metrizable topological space via infinite sequences of neighborhoods of diagonal. Then we prove that asymptotic average shadowing property implies topological chain transitivity.

2. Topological asymptotic average shadowing property

Denote by $\Sigma_{\mathscr{U}}$ the family of all sequences $\mathcal{E} = \{E_i\}_{i=0}^{\infty}$ of entourages in $\mathscr{U}$ with $E_0 = X \times X$, such that $E_{i+1} \subset E_i$ for all $i \in \mathbb{N}_0$. For a sequence $\mathcal{E} = \{E_i\}_{i=0}^{\infty} \in \Sigma_{\mathscr{U}}$, a map $f : X \to X$ and a sequence $\xi = \{x_0, x_1, \dots\}$ in X we define

$$\mathcal{A}_n(\xi, f, \mathcal{E}) = \mathcal{A}_n(\xi, f, \{E_i\}_{i=0}^{\infty}) = \inf\{\sum_{j=0}^{n} \frac{1}{2^{\sigma(j)}} | \quad (x_{j+1}, f(x_j)) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\}; \quad n \in \mathbb{N}$$

$$= \sum_{j=0}^{n} \inf\{\frac{1}{2^{\sigma(j)}} | \quad (x_{j+1}, f(x_j)) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^j\}; \quad n \in \mathbb{N},$$

and

$$\mathcal{A}_n(\xi, z, f, \mathcal{E}) = \mathcal{A}_n(\xi, z, f, \{E_i\}_{i=0}^{\infty}) = \inf\{\sum_{j=0}^{n} \frac{1}{2^{\sigma(j)}} | \quad (x_j, f^j(z)) \in E_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\},$$

where $\mathbb{N}_0^n$ is the set of all maps from $\{0, 1, \dots, n\}$ to $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Remark 2.1. Let $(X, \mathscr{U})$ be a compact uniform space and $f : X \to X$ be a continuous map. If $\mathcal{E} = \{E_i\}_{i=0}^{\infty} \in \Sigma_{\mathscr{U}}$ and $\xi = \{x_i\}_{i=0}^{\infty}$ be a sequence in X, then for any $m, n, k \in \mathbb{N}$ we have

(1) $0 \leq \mathcal{A}_n(\xi, f, \mathcal{E}) \leq n$;
(2) $\mathcal{A}_n(\xi, f, \mathcal{E}) \leq \mathcal{A}_{n+1}(\xi, f, \mathcal{E}) \leq \mathcal{A}_n(\xi, f, \mathcal{E}) + 1$;
(3) $\mathcal{A}_n(\xi, f, \{E_i\}_{i=0}^{\infty}) = \frac{1}{2^k}\mathcal{A}_n(\xi, f, \{E'_i\}_{i=0}^{\infty})$, where $E'_i = E_{i+k}$ for $i \geq 1$ and $E'_0 = X \times X$.
(4) $\mathcal{A}_n(\xi, f, \{E'_i\}_{i=0}^{\infty}) \leq \mathcal{A}_n(\xi, f, \{E_i\}_{i=0}^{\infty})$, where $E'_i = E_{ik}$;
(5) $\mathcal{A}_{m+n}(\xi, f, \mathcal{E}) = \mathcal{A}_m(\xi, f, \mathcal{E}) + \mathcal{A}_n(T^m(\xi), f, \mathcal{E})$, where $T : X^{\mathbb{N}_0} \to X^{\mathbb{N}_0}$ is the shift map.

Definition 2.2. [5] For $\mathcal{D} \in \Sigma_{\mathscr{U}}$, a *topological average $\mathcal{D}$-pseudo-orbit* of f is a sequence $\{x_i\}$ in X such that $\lim_{n\to\infty} \frac{1}{n}\mathcal{A}_n(\xi, f, \mathcal{D}) = 0$. Let $\mathcal{E} = \{E_i\} \in \Sigma_{\mathscr{U}}$. We say that the sequence $\{x_i\}$ is *$\mathcal{E}$-shadowed on average by some point* $z \in X$, if $\lim_{n\to\infty} \frac{1}{n}\mathcal{A}_n(\xi, z, f, \mathcal{E}) = 0$. We say that the map f has the *topological average shadowing property* **TASP**, if for every $\mathcal{E} = \{E_i\} \in \Sigma_{\mathscr{U}}$, there exists $\mathcal{D} = \{D_i\}_{i=0}^{\infty} \in \Sigma_{\mathscr{U}}$ such that every topological average $\mathcal{D}$-pseudo-orbit is $\mathcal{E}$-shadowed on average by some point of X.

If (X, d) is a compact metric space, then for any neighborhood U of Δ_X, we can find $\delta > 0$ such that $V_\delta^d \subset U$. On the other hand, every V_δ^d is a neighborhood of Δ_X. Moreover if $\{x_i\}$ is a topological average $\{V^d_{\frac{\delta}{2^i}}\}$-pseudo-orbit, then $\lim_{n\to\infty} \frac{1}{n}\mathcal{A}_n(\xi, f, \{V^d_{\frac{\delta}{2^i}}\}) = 0$ and there exists $N \in \mathbb{N}$ such that $\frac{1}{n}\mathcal{A}_n(\xi, f, \{V^d_{\frac{\delta}{2^i}}\}) < 1$ for $n \geq N$. One can easily check that

DEFINITION 2.3. An asymptotic-average-pseudo-orbit is a sequence $\{x_i\}$ in X such that for each $\mathcal{D} \in \Sigma_{\mathscr{U}}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \inf\{\sum \frac{1}{2^{\sigma(j)}} : \quad (fx_{j-1}, x_j) \in D_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\} = 0$$

We say that the sequence $\{x_i\}$ is asymptotically shadowed on average by some point $y \in X$, if for each $\mathcal{E} = \{E_i\} \in \Sigma_{\mathscr{U}}$ we obtain

$$\lim_{n\to\infty} \frac{1}{n} \inf\{\sum \frac{1}{2^{\sigma(j)}} : \quad (f^j(x), x_j) \in D_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\} = 0$$

DEFINITION 2.4. A map f is said to have the *asymptotic average shadowing property*, if every asymptotic-average-pseudo-orbit of f can be asymptotically shadowed on average by some point in X.

DEFINITION 2.5. A map f is said to have the *weak asymptotic average shadowing property*, if if for every $\mathcal{E} = \{E_i\} \in \Sigma_{\mathscr{U}}$ and any asymptotic pseudo-orbit $\xi = \{x_i\}$, there exists $z \in X$ such that $\lim_{n\to\infty} \frac{1}{n}\mathcal{A}_n(\xi, z, f, \mathcal{E}) = 0$.

The following example shows that **AASP** $\not\Rightarrow$ **TASP**.

EXAMPLE 2.6. Let $X = \{a, b, c\}$. Consider following subsets of $X \times X$

$U_0 = \Delta_X \cup \{(a, b), (b, a)\}$
$U_1 = \Delta_X \cup \{(a, b), (b, a), (a, c)\}$
$U_2 = \Delta_X \cup \{(a, b), (b, a), (a, c), (c, a)\}$
$U_3 = \Delta_X \cup \{(a, b), (b, a), (a, c), (b, c)\}$
$U_4 = \Delta_X \cup \{(a, b), (b, a), (a, c), (c, b)\}$
$U_5 = \Delta_X \cup \{(a, b), (b, a), (c, a)\}$
$U_6 = \Delta_X \cup \{(a, b), (b, a), (c, a), (b, c)\}$
$U_7 = \Delta_X \cup \{(a, b), (b, a), (c, a), (c, b)\}$
$U_8 = \Delta_X \cup \{(a, b), (b, a), (b, c)\}$
$U_9 = \Delta_X \cup \{(a, b), (b, a), (c, b)\}$
$U_{10} = \Delta_X \cup \{(a, b), (b, a), (b, c), (c, b)\}$
$U_{11} = \Delta_X \cup \{(a, b), (b, a), (b, c), (c, b), (a, c).\}$
$U_{12} = \Delta_X \cup \{(a, b), (b, a), (b, c), (c, b), (c, a)\}$
$U_{13} = \Delta_X \cup \{(a, b), (b, a), (b, c), (a, c).(c, a)\}$
$U_{14} = \Delta_X \cup \{(a, b), (b, a), (c, b), (a, c).(c, a)\}$

Then $\mathscr{U} = \{U_0, U_1, U_2, \dots, U_{14}, X\times X\}$ is a uniformity on X for which $\tau_{\mathscr{U}} = \{\emptyset, \{a, b\}, \{c\}, X\}$. Let $f : X \to X$ be a permutation defined by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then f is uniformly continuous. We show that f does not have the average shadowing property. Let $E_0 = X \times X$ and $E_i = U_0$ for all $i \geq 1$. Then $\mathcal{E} = \{E_i\} \in \Sigma_{\mathscr{U}}$. Put $x_0 = a$ and for each $i \in \mathbb{N}$ and $k \geq 0$ define

$$x_i = \begin{cases} f^i(a) & \text{if } 2^{2k} \leq i < 2^{2k+1}, \\ c & \text{if } 2^{2k+1} \leq i < 2^{2k+2} \end{cases}.$$

In other word

$$x_0, x_1, x_2, \dots = \underbrace{a, b}_{2}, \underbrace{c, c}_{2}, \underbrace{a, b, a, b}_{4}, \underbrace{c, c, \dots, c}_{8}, \underbrace{a, b, a, b, \dots, b}_{16}, \underbrace{c, c, \dots, c, c}_{32}, \dots.$$

Let $\xi = \{x_i\}$. Since any element of $\mathscr{U}$ contains (a, b) and (b, a), we obtain

$$\frac{1}{n} \inf\{\sum_{j=1}^{n} \frac{1}{2^{\sigma(j)}} : \quad (f(x_{j-1}), x_j) \in D_{\sigma(j)}, \sigma \in \mathbb{N}_0^n\} \leq \frac{2(k+1)}{n}$$
$$< \frac{2(k+1)}{2^k},$$

for $2^k \leq n < 2^{k+1}$ and arbitrary $\mathcal{D} = \{D_i\} \in \Sigma_{\mathscr{U}}$. That is ξ is an average $\mathcal{D}$-pseudo-orbit. For $2^k \leq n < 2^{k+1}$, we obtain

$$\frac{1}{n}\mathcal{A}_n(\xi, a, f, \mathscr{E}) = \frac{1}{n}\mathcal{A}_n(\xi, b, f, \mathscr{E}) \geq \frac{\sum_{i=0}^{k} 2^{2i-1}}{\sum_{i=0}^{2k} 2^i}.$$

Therefore ξ could not be $\mathcal{E}$-shadowed in average by any point in X. This implies that f does not have the topological average shadowing property. It is easy to show that f has the topological shadowing property and topological asymptotic average shadowing

PROPOSITION 2.7. *Let (X, f) be a dynamical system with uniform space. Then f has the asymptotically average shadowing property if f^k has for every $k \in \mathbb{N}$.*

Let $f : X \longrightarrow X$ be a homeomorphism of uniform compact Hausdorff space. Points $x, y \in X$ are called *proximal* if closure $\overline{\mathcal{O}((X, Y))}$ of the orbit of (x, y) under $f \times f$ intersects the diagonal $\Delta = \{(z, z) \in X \times X : z \in X\}$. Denote by $PR(X, f)$ the set of all pairs (x, y) where x and y are proximal.

THEOREM 2.8. *Let $(X, \mathscr{U})$ be a uniform space and $f : X \to X$ be continuous function. If f has the asymptotically average shadowing property, then $(x, y) \in PR \circ PR(X, f)$.*

THEOREM 2.9. *Let $(X, \mathscr{U})$ be a uniform space and $f : X \to X$ be continuous function. If f has the asymptotically average shadowing property, then $(x, y) \in PR \circ PR(X, f)$.*

COROLLARY 2.10. *Let $(X, \mathscr{U})$ be a compact uniform space and f be a continuous map from X onto itself. If f has asymptotic average shadowing property, then every point $x \in X$ is topological chain recurrent point.*

References

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