Mean Ergodic Shadowing: relation with other shadowing

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ABSTRACT. In this paper, we study mean ergodic shadowing property and obtain some results in relation with other partial shadowing. We show that mean ergodic shadowing implies $\overline{d}$-shadowing property, and any minimal system with mean ergodic shadowing does not have $\mathcal{F}_s$-shadowing property. In addition, by giving some examples, we show that the shadowing property and mean ergodic shadowing are different.

Keywords: Shadowing, mean ergodic shadowing, $\overline{d}$-shadowing

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1. Introduction

A pair (X, f), where (X, d) is a metric space and $f : X \to X$ is a continuous map is called a *topological dynamical system*. The shadowing theory is an important part of the global and stability theory of dynamical systems [?, 5]. The shadowing property means that near a pseudo-orbit (numerically computed orbit) there exists an exact orbit. In other words, numerical computations reflect the real dynamical behavior of f. Throughout the study (X, d) is a compact metric space. We use $\mathbb{Z}^+$ for the set of non-negative integers.

Given $\delta > 0$ a sequence $\xi = \{x_i\}_{i \in \mathbb{Z}^+} \subset X$ with the property

$$d(f(x_i), x_{i+1}) < \delta, \quad \forall i \in \mathbb{Z}^+$$

is called a δ-*pseudo-orbit* for f, and if

$$\underline{d}(\{i \geq 0 : d(f(x_i), x_{i+1}) < \delta\}) = 1,$$

where $\underline{d}(A)$ is the *lower density* of the set $A \subset \mathbb{Z}^+$ defined by

$$\underline{d}(A) = \liminf_{n \to \infty} \frac{|A \cap \{0, 1, \cdots, n-1\}|}{n},$$

is called a δ-*ergodic pseudo-orbit* of f [3]. If we replace lim inf with lim sup in the above formula we get $\overline{d}(A)$, the *upper density* of A. We say the set A has density zero if $\overline{d}(A) = 0$. A sequence $\xi = \{x_i\}_{i=0}^{\infty}$ is said to be ϵ-*shadowed* by a point $z \in X$ if

$$d(f^i(z), x_i) < \epsilon, \quad \forall i \geq 0.$$

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A map $f : X \to X$ is said to has *shadowing property* (POTP, for short) if for any $\epsilon > 0$ there exists $\delta > 0$ in which every δ-pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ can be ϵ-shadowed by some point in X.

DEFINITION 1.1. A δ-ergodic pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}^+}$ is said to be *ϵ-ergodic shadowed* by some point z in X if

$$\underline{d}(\{i \geq 0 : d(f^i(z), x_i) < \epsilon\}) = 1.$$

A dynamical system (X, f) has *ergodic shadowing property* if for any given $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ in which that any δ-ergodic pseudo-orbit of f can be ϵ-ergodic shadowed by some points in X [3].

A set $A \subset \mathbb{Z}^+$ is called *syndetic* if it has bounded gaps, i.e., there is $k > 0$ such that $A \cap \{i, i+1, \cdots, i+k-1\} \neq \emptyset$ for each $i \geq 0$. The family of syndetic sets is denoted by $\mathcal{F}_s$.

DEFINITION 1.2. If for any $\epsilon > 0$ there exists $\delta > 0$ such that for every δ-pseudo-orbit $\{x_i\}_{i\geq 0}$ there is a point $z \in X$ in which

$$\{i : d(f^i(z), x_i) < \epsilon\} \in \mathcal{F}_s,$$

then we say f has $\mathcal{F}_s$-shadowing property.

Analogously, we denote $\mathcal{F}_{\underline{d}}$ for the family of subsets of $\mathbb{Z}^+$ with positive lower density, and define $\mathcal{F}_{\underline{d}}$-shadowing property similarly.

DEFINITION 1.3. (X, f) has *$\underline{d}$-shadowing property* if for each $\epsilon > 0$ there exists $\delta > 0$ such that for every δ-ergodic pseudo-orbit $\{x_i\}_{i\geq 0}$ there is a point $z \in X$ such that

$$\underline{d}(\{i : d(f^i(z), x_i) < \epsilon\}) > 0.$$

Similarly, if $\overline{d}(\{i : d(f^i(z), x_i) < \epsilon\}) > \frac{1}{2}$ it is said *$\overline{d}$-shadowing property* (see [1, Definition 2.1]).

Recently, Das et al. [2] introduced a new type of average shadowing named mean ergodic shadowing and studied some aspects of it. We bring the following definition from [2].

DEFINITION 1.4. [2] A map f has *mean ergodic shadowing property* if for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ in which that any δ-ergodic pseudo-orbit of f can be ϵ-shadowed in average by some points in X . In other words, there exists $z \in X$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(z), x_i) < \epsilon.$$

It is well known that mean ergodic shadowing is a weaker form of ergodic shadowing property [2, Proposition 4.2].

1.1. Topological dynamical systems. A finite δ-pseudo-orbit $\{x_i\}_{i=0}^{b}$ is called a *δ-chain* from x_0 to x_b. A dynamical system (X, f) is called *chain transitive* if for any two points $x, y \in X$ and any $\delta > 0$ there exists a δ-chain from x to y. If for any $\delta > 0$ and any $x, y \in X$ there is $N > 0$ so that for every $n > N$ there is a δ-chain from x to y of length n, we say f is *chain mixing*. It is known that f is chain mixing if and only if f^n is chain transitive for every $n > 0$ [6].

For two nonempty open sets $U, V \subset X$, define $N(U, V) := \{n \in \mathbb{N} : f^n(U) \cap V \neq \emptyset\}$. In this regard, f is called *topological transitive* if for any two nonempty open sets $U, V \subset X$, $N(U, V) \neq \emptyset$. Moreover, if $\mathbb{N} \setminus N(U, V)$ is finite set f is called *topological mixing*.

2. Main results

First of all it is easy to see that mean ergodic shadowing is independent of choosing an equivalent metric. In fact, suppose d_1 and d_2 are two equivalent metric, and $\epsilon > 0$ is arbitrary. Take $0 < \epsilon_1 < \epsilon$ so that for each $x \in X$

$$B_{d_1}(x, \epsilon_1) \subset B_{d_2}(x, \epsilon_).$$

Note that this holds due to compactness of (X, d). Let $\delta_1 > 0$ be corresponding to ϵ_1 for f in definition of mean ergodic shadowing. Again choose $\delta_2 > 0$ in which

$$B_{d_2}(x, \delta_2) \subset B_{d_1}(x, \delta_1)$$

holds for every $x \in X$. Now, let $\{x_i\}_{i\geq 0}$ be a δ_2-ergodic pseudo-orbit for f with respect to d_2. So,

$$\overline{d}(\{i: \ d_2(f(x_i), x_{i+1}) \geq \delta_2\}) = 0.$$

By choosing δ_1 and δ_2 we also have

$$\overline{d}(\{i: \ d_1(f(x_i), x_{i+1}) \geq \delta_1\}) = 0.$$

Note that the latter set is subset of the former set. Hence, the sequence $\{x_i\}_{i\geq 0}$ is a δ_1-ergodic pseudo-orbit for f with respect to d_1. It follows there exists $z \in X$ such that $\overline{d}(E_1) < \epsilon_1$, where

$$E_1 = \{i: \ d_1(f^i(z), x_i) \geq \epsilon_1\}.$$

Eesily, by putting

$$E_2 = \{i: \ d_2(f^i(z), x_i) \geq \epsilon\},$$

we obtain $\overline{d}(E_2) < \epsilon$, because $E_2 \subset E_1$.

In the following, we show that mean ergodic shadowing is invariant of conjugacy.

Theorem 2.1. *Let (X, d_X) and (Y, d_Y) be two compact metric spaces, and let f and g be two continuous maps on (X, d_X) and (Y, d_Y) resp. If $h : X \to Y$ is a homeomorphism (conjugacy) between f and g, then $g = hofoh^{-1}$ has mean ergodic shadowing if and only if f has mean ergodic shadowing property.*

Proof. Suppose that f has mean ergodic shadowing, and $\epsilon > 0$ be given. Take $0 < \epsilon' < \epsilon$ by uniform continuity of h, i.e., $d_X(x, y) < \epsilon'$ implies $d_Y(h(x), h(y)) < \epsilon$. Suppose $\delta > 0$ is given for ϵ' by mean ergodic shadowing for f and δ' be given for δ by uniform continuity of h^{-1}, i.e. $d_Y(x, y) < \delta'$ implies $d_X(h^{-1}(x), h^{-1}(y)) < \delta$. Let $\{y_i\}_{i\in\mathbb{N}}$ be a δ'-ergodic pseudo-orbit for $g = hofoh^{-1}$, i.e. $\overline{d}(E) = 0$, where $E = \{i \in \mathbb{N} \mid d_Y(hofoh^{-1}(y_i), y_{i+1}) \geq \delta'\} \supset \{i \in \mathbb{N} \mid d_X(foh^{-1}(y_i), h^{-1}(y_{i+1})) \geq \delta\}$. This shows that $\{h^{-1}(y_i)\}_{i\in\mathbb{N}}$ is a δ-ergodic pseudo-orbit for f. So there is $z \in X$ by mean ergodic shadowing such that $\overline{d}(E') < \epsilon'$, where

$$\begin{aligned} E' =& \{i \in \mathbb{N} \mid d_X(f^i(z), h^{-1}(y_i)) \geq \epsilon'\} \supset \{i \in \mathbb{N} \mid d_Y(hof^i(z), y_i) \geq \epsilon\} \\ =& \{i \in \mathbb{N} \mid d_Y(g^i oh(z), y_i) \geq \epsilon\}. \end{aligned}$$

That is, $h(z)$ ϵ-shadowed $\{y_i\}_{i=0}^{\infty}$ in average, so g has mean ergodic shadowing. The remaining part is similar. □

In [2, Theorem 4.1] it is proved that in the presence of shadowing property the followings are equivalent for surjective dynamical system (X, f):

(1) f is totally transitive,
(2) f has almost average shadowing,
(3) f has mean ergodic shadowing,

(4) f has $\underline{d}$-shadowing.

We can also equalize specification property and topologically mixing with others, because by the above hypothesis totally transitivity implies chain mixing...

Corollary 2.2. *Under the hypothesis of [2, Theorem 4.1] the followings are equivalent:*

(1) f is totally transitive,
(2) f is topological mixing,
(3) f has almost average shadowing,
(4) f has mean ergodic shadowing,
(5) f has $\underline{d}$-shadowing,
(6) f has specification property.

We go on with the relation between $\mathcal{F}_s$-shadowing and mean ergodic shadowing property. We prove the following result:

Corollary 2.3. *The $\mathcal{F}_s$-shadowing property does not imply mean ergodic shadowing.*

Proof. Because every Morse-Smale diffeomorphism has shadowing property so clearly has $\mathcal{F}_s$-shadowing property, but they are not chain mixing and therefore by Corollary 2.2 do not have mean ergodic shadowing property. Refer to Example 3.1 as another example. □

Theorem 2.4. *Mean ergodic shadowing implies $\mathcal{F}_{\underline{d}}$-shadowing property.*

Proof. It is well known that $\underline{d}$-shadowing implies $\mathcal{F}_{\underline{d}}$-shadowing property. So, [2, Proposition 4.4] completes the proof. Alternatively, without loss of generality, let $0 < \epsilon < 1$ be given and $\delta > 0$ corresponds to ϵ in mean ergodic shadowing. Suppose $\{x_i\}_{i\geq 0}$ is a δ-psuedo-orbit for f, obviously, it is also a δ-ergodic psuedo-orbit for f. There exists $z \in X$ so that $\overline{d}(E) < \epsilon$, where $E = \{i \in \mathbb{N} \mid d(f^i(z), x_i) \geq \epsilon\}$. By the equality $\underline{d}(E^c) = 1 - \overline{d}(E)$ we obtain $\underline{d}(E^c) > 1 - \epsilon > 0$. That is f has $\mathcal{F}_{\underline{d}}$-shadowing property. □

In the following, we also prove that mean ergodic shadowing follows $\overline{d}$-shadowing property.

Proposition 2.5. *If f has mean ergodic shadowing, then it has $\overline{d}$-shadowing property.*

Proof. By inspiring of the proof of [?, Theorem 5], suppose $\epsilon > 0$ be given, and f has mean ergodic shadowing. Let $\delta > 0$ be correspond to $\frac{\epsilon}{2}$ in the definition of mean ergodic shadowing property for f. Take any δ-ergodic pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ so there exists $z \in X$ such that $\frac{\epsilon}{2}$-shadowed $\{x_i\}_{i=0}^{\infty}$ in average. Let $A = \{i : d(f^i(z), x_i < \epsilon)\}$ then we have

$$\begin{aligned}\frac{\epsilon}{2} > \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z), x_i) &\geq \limsup_{n\to\infty} \frac{\epsilon}{n}(n - \#A \cap \{0, \cdots, n-1\}) \\ &\geq \epsilon - \epsilon\overline{d}(A),\end{aligned}$$

so $\overline{d}(A) > \frac{1}{2}$. □

Corollary 2.6. *Every minimal dynamical system with mean ergodic shadowing does not have $\mathcal{F}_s$-shadowing property.*

Proof. It is well known that mean ergodic shadowing implies chain mixing. By [?, Theorem 16] every chain mixing minimal dynamical system with at least two points, does not have $\mathcal{F}_s$-shadowing property. So, every minimal dynamical system with mean ergodic shadowing does not have $\mathcal{F}_s$-shadowing property. □

QUESTION 2.7. *Is there any minimal system with mean ergodic shadowing property?*

COROLLARY 2.8. *The asymptotic average shadowing property (AASP, for short) implies mean ergodic shadowing, but not vice versa.*

PROOF. By [?, Theorem 4.3] AASP implies almost average shadowing property, and by [2, Proposition 4.3] almost average shadowing property in turn implies mean ergodic shadowing. However, the next example shows the converse does not hold. □

EXAMPLE 2.9. By the statement of Example 5.4 from [?] we construct the map φ on interval $[0,1]$ such that $\varphi(x) > x$ if and only if $x \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ It is easy to see that φ has mean ergodic shadowing, but by [4, Theorem 3.1, Remark 1] it does not have AASP.

Easily it can be shown that the analogous of [1, Theorem 4.2] holds in case of mean ergodic shadowing.

COROLLARY 2.10. *Any transitive sofic subshift σ with mean ergodic shadowing is mixing.*

PROOF. □

3. Examples

In this section, we bring two examples show that the shadowing property does not imply the mean ergodic shadowing property and vice versa. Also, we bring a class of maps without mean ergodic shadowing property.

EXAMPLE 3.1. The permutation of two points does not have the mean ergodic shadowing property, but has shadowing property.

PROOF. Let $X = \{a, b\}, f(a) = b, f(b) = a$. Without loss of generality suppose that $0 < \epsilon < \frac{d(a,b)}{4}$ is given and let $\delta > 0$ be arbitrary. If $\{m_i\}_{i=0}^{\infty}$ is a sequence of natural numbers defined by $m_i = 2^i$, it is obvious that $\{m_i\}$ has density zero. Now, let $\{x_i\}_{i=0}^{\infty} = \{a; a, b; b, a; a, b, a, b; b, a, \cdots\}$ be such that $x_{m_i} = x_{m_i+1}$ for each $i \geq 0$ and $\{x_{m_i+1}, \cdots, x_{m_{i+1}}\}$ is a finite δ-chain for each i. We show that the sequence $\{x_i\}$ cann't be ϵ-shadowed in average. Indeed, for $z = a$ we have

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{2^{2n}} \sum_{i=0}^{2^{2n}-1} d(f^i(z), x_i) &\geq \limsup_{n\to\infty} \frac{1}{2^{2n}} \sum_{i=2^{2n-2}+1}^{2^{2n-1}} d(f^i(z), x_i)\\ &= \limsup_{n\to\infty} \frac{2^{2n-2}}{2^{2n}}\, d(a,b) = \frac{1}{4}\, d(a,b) > \epsilon.\end{aligned}$$

Similarly, for $z = b$ we have

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{2^{2n}} \sum_{i=0}^{2^{2n}-1} d(f^i(z), x_i) &\geq \limsup_{n\to\infty} \frac{1}{2^{2n}} \sum_{i=2^{2n-1}+1}^{2^{2n}-1} d(f^i(z), x_i)\\ &= \limsup_{n\to\infty} \frac{2^{2n-1}}{2^{2n}}\, d(a,b) = \frac{1}{2}\, d(a,b) > \epsilon.\end{aligned}$$

Therefore, it doesn't have the mean ergodic shadowing. However, it is an easy exercise to check that the permutation of two points has the shadowing property. □

The following provides an example indicating mean ergodic shadowing does not imply the shadowing property.

Example 3.2. [?, Example 3.13] $f : \mathbb{S}^1 \ni e^{2\pi ix} \mapsto e^{2\pi ix^2} \in \mathbb{S}^1$ (where $x \in [0, 1)$ and $\mathbb{S}^1$ is the unit circle) has the AASP, and hence mean ergodic shadowing property by [?] and [2, Proposition 4.3], but obviously does not have the shadowing property.

Example 3.3. The circle rotations do not have mean ergodic shadowing.

Proof. □

4. Conclusion

In this paper we show that mean ergodic shadowing is a dynamical property, and study its relation with other type of partial shadowing.

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