Hypernormed Entropy

Khatereh Ghasemi[1,*], Javad Jamalzadeh [2]

[1]University of Sistan and Baluchestan, Zahedan, Iran

[2]Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

---

Abstract. In this paper, we introduce the notion of topological entropy on topological hypernormed hypergroup and provide some interesting examples. So, we obtain the fundamental properies of this entropy such as invariance under conjugation; Invariance under inversion;logarithmic law; Monotonicily for subflows; Continuity for direct limits.

Keywords: Hypernormed Entropy, Topological Hypergroup, Continuous Hypernormed

AMS Mathematics Subject Classification [2020]: 20N20, 22A30, 37B40

---

1. Introduction

Entropy is a tool to measure the amount of uncertainty in random events.The entropy has been applied in the information theory, physics, computer sciences, statistics, chemistry, biology, sociology, general systems theory and many other fields. The classical approach in the information theory was based on Shannon entropy. Shannon entropy of a probability distribution was studied in. Kolmogorov and Sinai used the Shannon entropy to define the entropy of measurable partitions and then they defined the entropy of dynamical systems. Kolmogorov-Sinai entropy is a useful tool in studying the isomorphism of dynamical systems. Adler, Konheim and McAndrew defined the topological entropy of a continuous self-map of a compact space. So, Bowen extended this notion to uniformly continuous self-maps of metric spaces. The notion of algebraic entropy was studied later by Weiss and Peters. Topological and Algebraic entropy were deeply studied [1–5]. Recently, Mehrpooya, Sayyari, Molaei proposed other definitions of Algebraic and Shannon entropies on commutative hypergroups.

In this paper, we introduce the notion of entropy on topological hypernormed hypergroup and prove Some interesting examples. So, we obtain the fundamental properties of this entropy such as Invariance under conjugation, Logarithmic Law, Monotonicity for subflows; Continuity for direct limits.

*Speaker. Email address: khatere.ghasemi@pgs.usb.ac.ir,

2. Preliminaries

The notion of hyperstructure, as a generalization of algebraic structure, was introduced by F. Marty at the 8th congress of Scandinavian Mathematicians in 1934 . One of the most important instances of hyperstructures is hypergroupoid. Let H be a nonempty set and $\mathcal{P}^\star(H)$ be the set of all non-empty subsets of H. A hyperoperation on H is a mapping $\circ : H\times H \to \mathcal{P}^\star(H)$. The pair $(H,\circ)$ is called a hypergroupoid. In the above definition, if A and$B$ are two non-empty subsets of H, then we define $A\circ B = \bigcup_{a\in A, b\in B} a\circ b; a\in A, b\in B$. A semihypergroup is a hypergroupoid $(H,\circ)$ such that:

$$\forall(a,b,c)\in H^3; a\circ(b\circ c) = (a\circ b)\circ c$$

A hypergroup is a semihypergroup $(H,\circ)$ such that: $\forall a\in H, a\circ H = H\circ a$. This condition is called the reproduction axiom.

A mapping $\varphi : (H,\circ)\to(H,\circ)$ is called good homomorphism if for every $x,y\in H$;

$$f(x\circ y) = f(x)\circ f(y).$$

We denote by $End(H)$ the set of good homomorphism of H.

Algebraic structures which also have a topology are useful in mathematics. In the same direction, some of mathematicians have studied the properties of hypergroupoids endowed with a topology. Ameri and Hoskova have defined and studied τ_u-topological hypergroup. Now we introduce a topology on $\mathcal{P}^\star(H)$.

LEMMA 2.1. *Let (H,τ) be a topological space. Then the family is consisting of all sets $V = \{U\in\mathcal{P}^\star(H) : U\subseteq V; V\in\tau\}$, is a basis for a topology τ_u on $\mathcal{P}^\star(H)$.*

DEFINITION 2.2. Let $(H,\circ)$ be a hypergroup and (H,τ) be a topological space. Then, the hyperoperation $\circ$ is said to be continuous if $\circ : H\times H\to\mathcal{P}^\star(H)$ is continuous. Here $H\times H$ and $\mathcal{P}^\star(H)$ are equipped with the product topology and τ_u , respectively.

Let $(H,\circ,\tau,\tau_u)$ be a topological hypergroup. We denote $End_c(H)$ the set of continuous good homomorphism.

3. Hypernormed entropy on topological hyper normed hypergroup

In this section, Let $(H,\circ,\tau,\tau_u)$ be a topological hypergroup and $\varphi\in End_c(H)$.

A mapping $\nu : \mathcal{P}^\star(H)\to\mathbb{R}^{\geq 0}$ is called hypernormed on H. For any $x\in H$; we define $\nu(x) = \nu(\{x\})$. If ν is a continuous on $(H,\circ,\tau,\tau_u)$, then $(H,\circ,\tau,\tau_u,\nu)$ is called topological hypernormed hypergroup.

DEFINITION 3.1. A mapping $\varphi : (H,\circ,\tau,\tau_u,\nu)\to(H',\circ',\tau',\tau'_u,\nu')$ is called contractive if

$$\nu'(\varphi(A)) \leq \nu(A), \tag{1}$$

for every $A\in\mathcal{P}^\star(H)$.

DEFINITION 3.2. A hypernormed ν on $\mathcal{P}^*(H)$ is called :

(a) subadditive, if $\nu(A\circ B)\leq\nu(A)+\nu(B)$ for every $A,B\in P^*(H)$.
(b) arithmetic, if for $x\in H$ there exists $c_x\in\mathbb{R}$ such that $\nu(x^n) = c_x logn$ for every $n\in\mathbb{N}$.
(c) ν is incresing respect to hyoeroperation such that $\nu(x),\nu(y)\leq\nu(x\circ y)$ for every $x,y\in H$.

Let $(H, \circ, \tau, \tau_u, \nu)$ be a topolpgical hypernormed hypergroup and
$\varphi \in End_c(H)$. consider the n-th φ-trajectory of $x \in H$

$$H_n(\varphi, x) = x \circ \varphi(x) \circ \cdots \circ \varphi^{n-1}(x)$$

and let

$$h_n(\varphi, x) = \nu(H_n(\varphi, x)).$$

now, we define

$$h(\varphi, x) = \limsup_n \frac{h_n(\varphi, x)}{n}.$$

LEMMA 3.3. *Let $(H, \circ, \tau, \tau_u, \nu)$ be a topological hypernormed hypergroup. Then $h(\varphi, x)$ is finite for every $x \in H$.*

Now, we define hypernormed entropy of $\varphi \in End_c(H)$, where $(H, \circ, \tau, \tau_u, \nu)$ is a topological hypernormed hypergroup,

$$h(\varphi) = \sup\{h(\varphi, x), x \in H\}.$$

In the following, we provide some partical examples of hypernormed entropy.

EXAMPLE 3.4. Let$(H, \circ, \tau, \tau_u)$ be a topological hypergroup and hypernormed
$\nu : \mathcal{P}^*(H) \to \mathbb{R}^{\geq 0}$ be a continuous and arthmetic.Then $h(I_H) = 0$ where
$I : H \to H$ is idetitical map.

EXAMPLE 3.5. Let $(\mathbb{R}, \circ, \tau, \tau_u)$ be a topological hypergroup with the standard topology, where $x \circ y = \{x, y\}$. Consider continuous hypernormed $\nu = ||.||$ and $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(x) = x + 1$ for every $x \in \mathbb{R}$.

$$H_n(\varphi, x) = \{x, x + 1, \cdots, x + n - 1\}$$

Then for every $x \in \mathbb{R}$ and so $h_n(\varphi, x) = n$, thus

$$h(\varphi, x) = \limsup_n \frac{n}{n} = 1$$

therefore $h(\varphi) = 1$.

4. Fundamental properties of hypernormed entropy

In this section, we define some of fundamental properties of entropy such as Invariance under conjugation, Logarithmic Law, Monotonicily for subflows; Continuity for direct limits.

THEOREM 4.1. (Existence of limit) *Let $(H, \circ, \tau, \tau_u, \nu)$ be a Topological hypernormed hypergroup. If ν is a subadditive hypernormed, then for every $x \in H$,* $\lim_n \sup \frac{h_n(\varphi,x)}{n}$ *exists.*

THEOREM 4.2. (monotonicily for factors) *Let $(H, \circ, \tau, \tau_u, \nu)$ and $(H', \circ', \tau', \tau'_u, \nu')$ be topological hypernormed hypergroup. And $\varphi \in End_c(H)$, $\psi \in End_c(H')$. If $\alpha : H \to H'$ is continuous contractive good epimorphism.*
Such that $\alpha \circ \psi = \psi \circ \alpha$. Then

$$h(\psi) \leq h(\varphi).$$

The following corollary is a direct consequence of Theorem 2.

COROLLARY 4.3. (Invariance under conjugation) *Let* $(H, \circ, \tau, \tau_u, \nu)$ *be a topological hypergroup and* $\varphi \in End_c(H)$. *If* $(H', \circ', \tau', \tau'_u, \nu')$ *is another topological hypernormed hypergroup and there exist a topological good isomorphism* $\alpha : H \to H'$ *such that* $\nu(a) = \nu'(\alpha(a))$ *for every* $a \in H$. *Then*

$$h(\varphi) = h(\alpha \circ \varphi \circ \alpha^{-1}).$$

LEMMA 4.4. *(Monotonicily for subflow)*
Let $(H, \circ, \tau, \tau_u, \nu)$ *be a topological hypernormed hypergroup and* $\varphi \in End_c(H)$. *If* G *is a* $\varphi-$*invariant* $(\varphi(G) = G)$ *subhypergroup. Then*

$$h(\varphi|_G) \leq h(\varphi)$$

One of the important properties of entropy is the law of logarithm. The defined entropy in lemma 3 has this property.

THEOREM 4.5. *Let* $(H, \circ, \tau, \tau_u, \nu)$ *be a topological hypernormed hypergroup and* $\varphi \in End_c(H)$. *If* ν *is increasing respect to hyperoperation* $\circ$, *then*

$$h(\varphi^k) = kh(\varphi),$$

for every $k \in \mathbb{N}$.

PROOF. Fix $k \in \mathbb{N}$.For every $x \in H$, we put

$$y = x \circ \varphi(x) \circ \cdots \circ \varphi^{k-1}(x)$$

we have

$$\begin{aligned} h_n(\varphi^k, y) &= \nu(y \circ \varphi^k(y) \circ \cdots \circ \varphi^{n-1}(\varphi^k(x))) \\ &= h_{nk}(\varphi, x) \end{aligned}$$

for every $n \in \mathbb{N}$. So

$$\begin{aligned} h(\varphi^k) &\geq h(\varphi^k, y) \\ &= \limsup_n \frac{h_n(\varphi^k, y)}{n} \\ &= k \limsup_n \frac{h_{nk}(\varphi, x)}{kn} \\ &= kh(\varphi, x) \end{aligned}$$

consequently, $h(\varphi^k) \geq kh(\varphi)$.

Now,we prove the converse inequality. Since ν is increasing respect to $\circ$, then for every $n \in \mathbb{N}, x \in H$, we finde that deduce

$$\begin{aligned} h_{nk}(\varphi, x) &= v(x \circ \varphi(x) \circ \cdots \circ \varphi^{nk-1}(x)) \\ &\geq V(x \circ \varphi^k(x) \circ \cdots \circ (\varphi^k)^{n-1}(x)) \\ &= V(H_n(\varphi^k, x)) \\ &= h_n(\varphi^k, x) \end{aligned}$$

Thus

$$\begin{aligned} h(\varphi, x) &= \limsup_n \frac{h_{nk}(\varphi, x)}{nk} \\ &\geq \frac{1}{k} \limsup_n \frac{h_n(\varphi^k, x)}{n} \end{aligned}$$

$$= \quad \frac{h(\varphi^k, x)}{k}.$$

There fore

$$kh(\varphi) \geq h(\varphi^k),$$

this complate the proof.

□

THEOREM 4.6. *(ciontinuity for direct limits) Let* $(H, \circ, \tau, \tau_u, \nu)$ *be a direct limit of* $\varphi -$ *invariant subhypergroups* $\{H_i; i \in I\}$. *Then*

$$h(\varphi) = \sup\{h(\varphi|_{H_i}); i \in I\}.$$

5. Conclusion

In this paper, we introduce the notion of entropy on topological hypernormed hypergroup and prove Some interesting examples. So, we obtain the fundamental properties of this entropy such as Invariance under conjugation, Logarithmic Law, Monotonicity for subflows; Continuity for direct limits.

Acknowledgement

Finally, thank you to the conference organizers.

References

1. B. Davvaz, V. Leorena-Fortea, *Hyperring Theory and applications*, International Academic Press, Usb, 2007.
2. D. Dikranjan, A. Giordano Bruno, *The connection between topological and algebraic entropy*, Topol. Appl. **159** (13) (2012) 2980–2989.
3. D. Heidari, S.M.S Modarres, B. Davvaz, *Topological hypergroups in the sense of Marty*, Commun. Algebra **42** (2014) 4712–4721.
4. S. Hoskova-Mayerova, *Topological hypergroupoids*, Comput. Math. Appl. **64** (2012) 2845–2849.
5. A. Mehrpooya, Yamin Sayyari, M.R. Molaei, *Algebraic and Shannonen tropies of commutative hypergroups and their connection with in formation and permutation entropies and with calculation of entropy for chemical algebras*, Soft Computing **23** (24) (2019), 13035–13053