Gradient Ricci harmonic-Bourguignon solitons on multiply warped products

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Abstract. In this paper, we study certain conditions that a multiply warped product could be a gradient Ricci harmonic-Bourguignon soliton. Also, we obtain some conditions that potential function be constant and consequently multiply warped product be a harmonic-Einstein manifold.

Keywords: Gradient Ricci harmonic-Bourguignon soliton, Multiply warped product

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1. Introduction

Let (M, g) and (N, h) be complete Riemannian manifolds and $\varphi : M \longrightarrow N$ be a critical point of the energy integral $E(\varphi) = \int_M |\nabla\varphi|^2 dv_g$, where N is isometrically embeded in $\mathbb{R}^d$, $d \geq n$. By a one parameter family of Riemannian metrics $(g(x, t), \varphi(x, t))$, $t \in [0, T)$ and a family of smooth functions $\varphi(x, t)$, a Ricci harmonic-Bourguignon flow on manifold M is defined as

$$\begin{aligned} \frac{\partial}{\partial t} g(x,t) &= -2\mathrm{Ric}(x,t) + 2\rho R(x,t) + 2\alpha\nabla\varphi(x,t) \otimes \nabla\varphi(x,t), \\ \frac{\partial}{\partial t} g(x,t) &= \tau_g \varphi(x,t). \end{aligned}$$

Here α and ρ are positive constants, Ric is the Ricci tensor of M, R is the scalar curvature, and $\tau_g\varphi$ is the intrinsic Laplacian of φ which denotes the tension field of map φ [?]. The system $(M, g, X, \lambda, \rho, \varphi)$ is said to define a Ricci harmonic-Bourguignon soliton (RHBS for short) when it satisfies in the following coupled equation

$$\begin{aligned} \mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g + \rho R g + \alpha\nabla\varphi \otimes \nabla\varphi, \\ \tau_g\varphi - \mathcal{L}_X \nabla\varphi = 0, \end{aligned}$$

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*Speaker. Email address: s4004196001@edu.ikiu.ac.ir, where λ, α and ρ are constants, R is scalar curvature and φ is a smooth function $\varphi : (M, g) \to (N, h)$ where M and N are static Riemannian manifolds. In definition of RHBS if $X = \nabla f$, which f is a smooth function on M, then we say M is a gradient Ricci-harmonic-Bourguignon soliton (GRHBS for short). In this case we have

$$\begin{aligned} \mathrm{Ric} + \mathrm{Hess} f - \rho R g - \alpha \nabla\varphi \otimes \nabla\varphi &= \lambda g \\ \tau_g \varphi - < \nabla\varphi, \nabla f > &= 0. \end{aligned} \tag{1}$$

The function f is called the potential. The GRHBS is steady, expanding or shrinking if $\lambda = 0$, $\lambda < 0$ or $\lambda > 0$ respectively. Actually gradient Ricci solitons are a particularly interesting family of Ricci solitons. These arise as self similar solutions of the Ricci flow under certain conditions. If in (1), $\alpha = 0$ or φ is a constant function, then it defines gradient Ricci Bourguignon soliton and if $\rho = 0$, then it defines gradient Ricci-harmonic soliton. For more study about these kind of solitons see [**?**, 1].

Let (B^r, g_B) and $(F_s^{m_s}, g_{F_s})$ be semi-Riemannian manifolds for $1 \le s \le l$ and $M = B \times F_1 \times F_2 \times ... \times F_l$ be an n-dimentional semi-Riemannian manifold. Let $b_s : B \longrightarrow (0, \infty)$ be positive smooth functions for $1 \le s \le l$. The multiply warped product manifold is the product manifold $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_l} F_l$ endowed with the metric tensor $g = \pi^*(g_B) \oplus (b_1 \circ \pi)^2 \sigma_1^*(g_{F_1}) \oplus ... \oplus (b_l \circ \pi)^2 \sigma_l^*(g_{F_l})$, where π and σ are the natural projections on B and F_i, respectively [**?**]. In [**?**], Fatma Karaca studied about the necessary conditions for a multiply warped product to be a gradient Ricci-harmonic soliton. Till now so many different results have been found about the sufficient conditions for a multiply warped product and also doubly warped product to be different kinds of Ricci soliton such as gradient Ricci solitons, gradient harmonic solitons and gradient Yamabe solitons. Motivated by those work we studied some conditions that a multiply warped product could be a GRHBS.

2. Main results

We shall denote ∇, ∇_B and ∇_{F_s}; Ric, Ric_B and Ric_{F_s}; Δ, Δ_B and Δ_{F_s}; the Levi-civita connections, the Ricci tensors and the Laplacians of M, B and F respectively. Here is our main results. First of all we want to characterize the harmonic map φ by means of the potential function f. For this aim we obtain:

Proposition 2.1. *Let $(M = B \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_l} F_l, g, f, \varphi, \lambda, \rho)$ be a GRHBS on a multiply warped product with non-constant harmonic function φ, then for a neighborhood V around $(p, q_1, ..., q_l)$, it can be shown like $\varphi = \varphi_B \circ \pi$ or $\varphi = \varphi_{F_s} \circ \sigma_s$ for $1 \le s \le l$ iff $h = h_B \circ \pi$.*

Now we want to know the structure of Ric_B and Ric_{F_s} for a multiply warped product which could be a GRHBS.

Theorem 2.2. *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times ... \times_{b_l} F_l$ be a multiply warped product manifold. M is a GRHBS iff*
1) For $\varphi = \varphi_B \circ \pi$ we have

$$\begin{cases} \mathrm{Ric}_B - \sum_{s=1}^{l} \frac{m_s}{b_s} \mathrm{Hess}_B(b_s) + \mathrm{Hess}_B h_B - \rho R g_B - \alpha \nabla_B \varphi_B \otimes \nabla_B \varphi_B = \lambda g_B, \\ \Delta_w \varphi_B = 0 \;\; in\ B. \end{cases} \tag{2}$$

Here $\Delta_w = \Delta - < \nabla, \nabla w >$, $w = h - \sum_{s=1}^{l} m_s log(b_s)$. F_s is Einstein manifold for all $1 \le s \le l$ with $\mathrm{Ric}_{F_s} = \mu_s g_{F_s}$, where

$$\mu_s = \lambda b_s^2 + b_s(\Delta_B b_s) + (m_s - 1)\|\nabla_B b_s\|^2 + b_s \nabla_B h_B(b_s)$$

$$+\sum_{k=1,k\neq s}^{l}\frac{m_k}{b_k}g_B(\nabla_B b_s,\nabla_B b_k)b_s+\rho R b_s^2.$$

2) For $\varphi=\varphi_{F_s}\circ\sigma_s$, we have

$$\mathrm{Ric}_B-\sum_{s=1}^{l}\frac{m_s}{b_s}\mathrm{Hess}_B(b_s)+\mathrm{Hess}_B h_B-\rho R g_B=\lambda g_B, \tag{3}$$

and F_s are harmonic-Einstein manifolds so that

$$\begin{cases}\mathrm{Ric}_{F_s}-\alpha\nabla_{F_s}\varphi_{F_s}\otimes\nabla_{F_s}\varphi_{F_s}=\mu_s g_{F_s}\\ \Delta_{F_s}\varphi_{F_s}=0\end{cases} \tag{4}$$

which for all $1\leq s\leq l$, we have

$$\begin{aligned}\mu_s=&\lambda b_s^2+\rho R b_s^2+b_s(\Delta_b b_s)+(m_s-1)\|\nabla_B b_s\|^2+b_s\nabla_B h_B(b_s)\\ &+\sum_{k=1,k\neq s}^{l}\frac{m_k}{b_k}g_B(\nabla_B b_s,\nabla_B b_k)b_s.\end{aligned}$$

Now, we give some results for the potential function and harmonic function with use of maximum principle and give some conditions that cause the multiply warped product M to be a harmonic-Einstein manifold.

Theorem 2.3. *Suppose that $M=B\times_{b_1}F_1\times_{b_2}F_2\times...\times_{b_l}F_l$ is a GRHBS on multiply warped product with non-constant harmonic map φ.*
1) For $1\leq s\leq l$, either $\varphi=\varphi_B\circ\pi$ or $\varphi=\varphi_{F_s}\circ\sigma_s$, it is a constant function and M is a gradient Ricci-Bourguignon soliton if φ_B or φ_{F_s} has the maximum or minimum in B and F_s.
2) For $\lambda,\rho,R\geq 0$, h_B reaches the maximum or minimum in B and $h=h_B\circ\pi$ is a constant map. Therefore M is a harmonic-Einstein manifold.

We consider a GRHBS on a multiply warped product with harmonic map $\varphi=\varphi_B\circ\pi$ when the base manifold is conformal to an n-dimentional semi-Euclidean space, invariant under the action of an $(r-1)$-dimentional tranlation group. Let $M=(\mathbb{R}^r,\phi^{-2}g_{\mathbb{R}})\times_{b_1}F_1\times_{b_2}F_2\times...\times_{b_l}F_l$ be a multiply warped product endowed with the metric tensor

$$g=\frac{1}{\phi^2}g_{\mathbb{R}}+b_1^2g_{F_1}+...+b_l^2g_{F_l}, \tag{5}$$

which here $g_{\mathbb{R}}$ is the canonical semi-Riemannian metric and φ is the conformal factor. Actually, we have the semi-Riemannian metric $(g_{\mathbb{R}})_{i,j}=\epsilon_i\delta_{i,j}$ in the coordinates $x=(x_1,...,x_r)$ of $\mathbb{R}^r$, $\epsilon_i=\pm1$. We consider the function $\xi(x_1,...,x_r)=\sum_{i=1}^{r}\beta_i x_i$, where $\beta_i\in\mathbb{R}$. We take $\varphi=\varphi_B\circ\pi$ for the next theorem.

Theorem 2.4. *Let $M=\mathbb{R}^r\times_{b_1}F_1\times_{b_2}F_2\times...\times_{b_l}F_l$ be a multiply warped product with non-constant harmonic map φ and $b_s=b_s\circ\xi$, $h=h\circ\xi$, $\varphi=\varphi\circ\xi$, $\phi=\phi\circ\xi$ defined in $(\mathbb{R}^r,\phi^{-2}g_{\mathbb{R}})$ endowed with the metric (5), then M is a GRHBS iff the functions b_s, h, φ and ϕ satisfy in the following equations:*

$$(r-2)\frac{\phi''}{\phi}-\sum_{s=1}^{l}m_s\frac{b_s''}{b_s}-2\sum_{s=1}^{1}m_s\frac{b_s'}{b_s}\frac{\phi'}{\phi}+h''+2\frac{\phi'}{\phi}h'-\alpha(\varphi)^2=0, \tag{6}$$

$$\left(\frac{\phi^{''}}{\phi}-(r-1)\left(\frac{\phi^{'}}{\phi}\right)^2+\sum_{s=1}^{l}m_s\frac{b_s^{'}}{b_s}\frac{\phi^{'}}{\phi}-\frac{\phi^{'}}{\phi}h^{'}\right)\|\beta\|^2=\frac{\lambda+\rho R}{\phi^2},\tag{7}$$

$$\begin{aligned}&\left(\frac{b_s^{''}}{b_s}-(r-2)\frac{\phi^{'}}{\phi}\frac{b_s^{'}}{b_s}+(m_s-1)\left(\frac{b_s^{'}}{b_s}\right)^2+\sum_{k=1,k\neq s}^{l}\left(m_k\frac{b_s^{'}}{b_s}\frac{b_k^{'}}{b_k}\right)+\frac{b_s^{'}}{b_s}h^{'}\right)\|\beta\|^2\\&=\frac{\mu_s}{b_s^2\phi^2}-\frac{\lambda+\rho R}{\phi^2},\end{aligned}\tag{8}$$

$$\left(\varphi^{''}-(r-2)\frac{\phi^{'}}{\phi}\varphi^{'}+\sum_{s=1}^{1}m_s\frac{b_s^{'}}{b_s}\varphi^{'}-\varphi^{'}h^{'}\right)\|\beta\|^2=0.\tag{9}$$

Now, we consider a GRHBS on a multiply warped product with harmonic map $\varphi=\varphi_{F_s}\circ\sigma_s$ when the base manifold and fibers are conformal to r-dimentional and m_i-dimentional semi-Euclidean spaces, invarient under the action or $(r-1)$-dimentional and (m_i-1)-dimentional translation groups for $1\leq i\leq l$, respectively. Let $M=(\mathbb{R}^r,\phi^{-2}g_{\mathbb{R}})\times_{b_1}(\mathbb{R}^{m_1},\tau_1^{-2}g_{\mathbb{R}})\times_{b_2}(\mathbb{R}^{m_2},\tau_2^{-2}g_{\mathbb{R}})\times...\times_{b_l}(\mathbb{R}^{m_l},\tau_l^{-2}g_{\mathbb{R}})$ be a multiply warped product endowed with the metric tensor

$$g=\frac{1}{\phi^2}g_{\mathbb{R}}+b_1^2\frac{1}{\tau_1^2}g_{\mathbb{R}}+...+b_l^2\frac{1}{\tau_l^2}g_{\mathbb{R}},\tag{10}$$

here ϕ and τ_i for $1\leq i\leq l$ are the conformal factors of base and fibers, respectively. We define function ζ_s for nonzero arbitrary vectors $a=(a_{r+1},...,a_{r+m_s})$ and $y=(x_{r+1},...,x_{r+m_s})$ as follows

$$\zeta_s(x_{r+1},...,x_{r+m_s})=a_{r+1}x_{r+1},...,a_{r+m_s}x_{r+m_s}.$$

Theorem 2.5. *Let $M=\mathbb{R}^r\times_{b_1}\mathbb{R}^{m_1}\times_{b_2}\mathbb{R}^{m_2}\times...\times_{b_l}\mathbb{R}^{m_l}$ be a multiply warped product with non-constant harmonic map $\varphi=\varphi_{F_s\circ\sigma_s}$ and $b_s=b_s\circ\xi$, $h=h\circ\xi$, $\phi=\phi\circ\xi$, $\varphi=\varphi\circ\zeta_s$ defined in $(\mathbb{R}^r,\phi^{-2}g_{\mathbb{R}})$ and $(\mathbb{R}^{m_s},\tau_s^{-2}g_{\mathbb{R}})$ for $1\leq s\leq l$ with the metric tensor (10), then M is a GRHBS iff the functions b_s, h, ϕ, φ satisfy*

$$(r-2)\frac{\phi^{''}}{\phi}-\sum_{s=1}^{l}m_s\frac{b_s^{''}}{b_s}-2\sum_{s=1}^{l}m_s\frac{b_s^{'}}{b_s}\frac{\phi^{'}}{\phi}+h^{''}+2\frac{\phi^{'}}{\phi}h^{'}=0,\tag{11}$$

$$\left(\frac{\phi^{''}}{\phi}-(r-1)(\frac{\phi^{'}}{\phi})^2+\sum_{s=1}^{l}m_s\frac{b_s^{'}}{b_s}\frac{\phi^{'}}{\phi}-\frac{\phi^{'}}{\phi}h^{'}\right)\|\beta\|^2=\frac{\lambda+\rho R}{\phi^2},\tag{12}$$

$$\begin{aligned}&\left(b_sb_s^{''}\phi^2-(r-2)\phi\phi^{'}b_sb_s^{'}+(m_s-1)\phi^2(b_s^{'})^2+\sum_{k=1,k\neq s}^{l}\left(m_k\phi^2\frac{b_k^{'}}{b_k}b_sb_s^{'}\right)+b_sb_s^{'}\phi^2h^{'}\right)\|\beta\|^2\\&+(\lambda+\rho R)(b_s)^2=[\tau_s\tau_s^{''}-(m_s-1)(\tau_s^{'})^2]\|a\|^2,\end{aligned}\tag{13}$$

$$(m_s-2)\frac{\tau_s^{''}}{\tau_s}-\alpha(\varphi^{'})^2=0,\tag{14}$$

$$(\varphi^{''}\tau_s^2-(m_s-2)\tau_s\tau_s^{'}\varphi^{'})\|a\|^2=0.\tag{15}$$

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