Study of qualitative behavior of a new coronavirus disease model

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Abstract. The aim of this manuscript is to discuss the dynamics of a coronavirus disease 2019 (COVID–19) model. We first prove the positivity and boundedness of solution of the proposed COVID–19 model. Thence, we determine the equilibrium points and discuss the stability analysis of the model. In continuation, we show that the equilibrium points are locally asymptotically stable. We apply the nonstandard finite difference (NSFD) scheme to study the dynamic behaviors COVID–19 model. In order to the efficiency and accuracy of the proposed NSFD, some numerical results are presented.

Keywords: Coronavirus model, positivity, boundedness, stability analysis, nonstandard finite difference scheme.

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1. Introduction

Over the years, mathematical modelling proves its ability to obtain more understanding of the dynamics disease models in the community. These models can help the researchers to understand more about the spread process of virus that may turn into a pandemic situation and may predict the conditions that will show the continuation or end of these infections. In March 2020, the COVID–19 disease begins to spread throughout of the world which is originated from Wuhan in China causing a global fear and devastating effect which conclude the governments and scientists to find a suitable cure [1, 2]. This virus can mainly transmitted through the droplets of an infection person that can spread when the person coughs, sneezes, or even while talking. The cases for the COVID–19 have been increasing since the beginning of 2020 such that now causing more than 230

*Speaker. Email address: namjoo@vru.ac.ir million infected persons and approximately 5 million deaths with a rate recovery 90 percent and 10 percent death rate. Due to the above reasons, scientists have been working extensively overall the last two years [3, 4]. In many cases, mathematical modelling of a COVID–19 model can be described by a nonlinear autonomous initial value problem. Since analytical solution a few numbers of these equations cannot obtained, hence, various numerical methods were constructed to solve such equations. In this research, in order to approximate the solution of the proposed COVID–19 model, we are going to construct an efficient NSFD scheme. An appropriate mathematical model for the COVID–19 at time t can be written as follows

$$\begin{cases} s'(t) = \gamma r(t) - \alpha s(t)i(t) - \mu_s s(t) + \mu^*, \\ i'(t) = \alpha s(t)i(t) - \beta i(t) - \mu_i i(t), \\ r'(t) = \beta i(t) - \gamma r(t) - \mu_r r(t), \\ s(0) = s_0,\ i(0) = i_0,\ r(0) = r_0. \end{cases} \tag{1}$$

In this model, the total population individuals at each time t is divided into three groups. Here, $s(t)$ is the number of susceptible group at time t, $i(t)$ is the number of infected group at time t and $r(t)$ denotes the recovered group at time t. Also, moving from the susceptible group to the infected group occurs at a rate α and infected groups are supposed to recover at a constant rate β. Moreover, the recovered individuals can again return to the susceptible group at a constant rate of γ. Here the parameters $\mu_s,\ \mu_i$ and μ_r denote death rates of the susceptible, infected and recovered groups, respectively and the parameter μ^* presents rates of birth the susceptible, infected and recovered groups. The organization of the manuscript is as follows. In Section 2, we prove positivity and boundedness of the solution model of (1). Section 3 deals with stability analysis of COVID–19 model. In Section 4, we construct an efficient NSFD scheme for the COVID–19 model (1). The numerical results are obtained by the NSFD scheme, show the efficiency of the NSFD scheme.

2. Positivity and boundedness of solutions

In this part, we are going to show that the state variables are nonegative and bounded that describes the COVID–19 model meaningful. First, we want to show that the solutions $s(t)$, $i(t)$ and $r(t)$ of the model (1), when they exist, are positive for all $t \geq 0$ with nonegative initial conditions.

THEOREM 2.1. *Consider the initial conditions as given in* (1). *Then the solutions* (s, i, r) *are positive for all time* $t \geq 0$.

PROOF. Since the sr-coordinate plane is invariant under the flows of system, this implies that $i(t) > 0$ for all $t \geq 0$. Let $A = \{t \geq 0 | r(t) < 0\}$, we will show that $A = \emptyset$. Suppose that $A \neq \emptyset$ and let $t_0 = \inf(A)$. Since $r(0) > 0$, so $t_0 > 0$. Now the continuity of r implies that $r(t_0) = 0$ and by the third equation of system (1), $r'(t_0) = \beta i(t_0) > 0$. Hence, there is $\varepsilon > 0$ such that $r(t) > 0$ for all $t \in (t_0 - \varepsilon,\ t_0 + \varepsilon)$. Consequently, $r(t) \geq r(t_0) > 0$, for all time $t \in (t_0,\ t_0 + \varepsilon)$ which contradicts $t_0 = \inf(A)$. By a similar argument, we can show that $s(t) \geq 0$, for all $t \geq 0$. □

In order to prove boundedness of the solutions of the system (1), we first state the following proposition.

PROPOSITION 2.2. *Let $K(t) : [0, +\infty) \longrightarrow \mathbb{R}$ be a derivative function such that $K(t) \geq 0$ for all $t \geq 0$. If $\alpha > 0$, $\beta \in \mathbb{R}$, such that $K'(t) + \alpha K(t) \leq \beta$, for all $t \geq 0$, then $K(t) \leq K(0) + \frac{\beta}{\alpha}$.*

LEMMA 2.3. *All the solutions $(s(t), i(t), r(t))$ of the system (1) are bounded.*

PROOF. Set $K(t) = s(t) + i(t) + r(t)$ and suppose that $m = \min\{\mu_s,\ \mu_i,\ \mu_r\}$. Hence $K(t) + mK'(t) \leq \mu^*$. It follows from proposition 2.2 that $s(t) + i(t) + r(t) \leq s(0) + i(0) + r(0) + \frac{\mu^*}{m}$. This shows that the solutions s, i, r of model (1) are bounded. □

3. Stability analysis for the COVID–19 model

The equilibrium points of the COVID–19 model (1) are given by $E_1 = (\frac{\mu^*}{\mu_s}, 0, 0)$ and $E_2 = (\frac{\mu_i+\beta}{\alpha}, i^*, \frac{\beta i^*}{\gamma+\mu_r})$, where $i^* = \frac{\mu_s(\frac{\mu_i+\beta}{\alpha})-\mu^*}{\frac{\gamma\beta}{\gamma+\mu_r}-\mu_i-\beta}$.

THEOREM 3.1. *The system (1) is*

(i) locally asymptotically stable at the equilibrium point E_1 if and only if $\alpha\frac{\mu^*}{\mu_s} - \beta - \mu_i < 0$.

(ii) locally asymptotically stable at the equilibrium point E_2 if and only if $i^* > 0$.

PROOF. The Jacobian matrix of system (1) corresponding to any equilbrium point (s_1, i_1, r_1) can be written as

$$J(s_1, i_1, r_1) = \begin{bmatrix} -\alpha i_1 - \mu_s & -\alpha s_1 & \gamma \\ -\alpha i_1 & \alpha s_1 - \beta - \mu_i & 0 \\ 0 & \beta & -\gamma - \mu_r \end{bmatrix}.$$

The Jacobian matrix of (1) at the equilibrium point E_1 is obtained as given below

$$J(E_1) = \begin{bmatrix} -\alpha\dfrac{\mu^*}{\mu_s} - \mu_s & -\alpha\frac{\mu^*}{\mu_s} & \gamma \\ 0 & \alpha\dfrac{\mu^*}{\mu_s} - \beta - \mu_i & 0 \\ 0 & \beta & -\gamma - \mu_r \end{bmatrix}.$$

The corresponding eigenvalues are $\lambda_1 = -\alpha\frac{\mu^*}{\mu_s} - \mu_s$, $\lambda_2 = \alpha\frac{\mu^*}{\mu_s} - \beta - \mu_i$ and $\lambda_3 = -\gamma - \mu_r$. Therefore the equilibrium point E_1 is locally asymptotically stable if and only if $\alpha\frac{\mu^*}{\mu_s} - \beta - \mu_i < 0$. At the equilibrium point E_2 the Jacobian matrix is given

$$J(E_2) = \begin{bmatrix} -\alpha i^* - \mu_s & -\mu_i - \beta & \gamma \\ \alpha i^* & 0 & 0 \\ 0 & \beta & -\gamma - \mu_r \end{bmatrix}.$$

Hence, we obtain that the characteristic equation can be presented in the following form

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3, \tag{2}$$

where

$a_1 = \alpha i^* + \mu_s + \gamma + \mu_r$, $a_2 = (\alpha i^* + \mu_s)(\gamma + \mu_r) + \alpha i^*(\mu_i + \beta)$, $a_3 = \alpha i^*(\mu_i + \beta)(\gamma + \mu_r) - \gamma\alpha i^*\beta$.

Using the Routh-Hurwitz criteria, all roots of Eq. (2) have negative real parts if and only if

$$a_1 > 0,\ a_2 > 0,\ a_3 > 0,\ a_1a_2 - a_3 > 0. \tag{3}$$

However, it can be shown that the conditions given by Eq. (3) will fulfill if $i^* > 0$. Hence, the equilibrium point E_2 is locally asymptotically stable if and only if $i^* > 0$. □

4. A NSFD scheme for the COVID–19 model

In this section, we are going to develop an explicit numerical scheme using NSFD scheme which were firstly proposed by Mickens for an initial value problem. Many applications are available in literature using NSFD scheme [5]. In order to introduce the general aspect of a NSFD scheme consider the following autonomous initial value problem

$$X'(t) = f(X(t)), \quad X(0) = X_0, \quad t \in [0, t_f]. \tag{4}$$

Suppose that a discretization $t_k = kh$ is given. A NSFD scheme for the problem (4) is constructed by the following two steps.

(i) The first order deviation in the problem (4) at the k-th time step can be replaced by a discrete form $X'(t_k) \approx \frac{X_{k+1}-X_k}{\phi(h)}$, where X_k is an approximation of the exact solution $X(t_k)$ and moreover the denominator function $\phi(h)$ has to satisfy the condition $\phi(h) = h + O(h^2)$ with $0 < \phi(h) < 1$.

(ii) The nonlinear and linear terms in the right–hand–side equation have to replace by nonlocal discrete approximations. According to the Mickens rules, a NSFD scheme for the proposed COVID–19 model (1) can be written as

$$\begin{cases} \dfrac{s_{k+1}-s_k}{\phi_1} = \gamma r_k - \alpha s_{k+1} i_k - \mu_s s_{k+1} + \mu^*, \\ \dfrac{i_{k+1}-i_k}{\phi_2} = \alpha s_{k+1} i_k - \beta i_{k+1} - \mu_i i_{k+1}, \\ \dfrac{r_{k+1}-r_k}{\phi_3} = \beta i_{k+1} - \gamma r_{k+1} - \mu_r r_{k+1}, \end{cases} \tag{5}$$

where the denominator functions are defined as

$$\phi_1(h) = \frac{e^{\mu_s h}-1}{\mu_s}, \qquad \phi_2(h) = \frac{e^{(\beta+\mu_i)h}-1}{\beta+\mu_i}, \qquad \phi_3(h) = \frac{e^{(\gamma+\mu_r)h}-1}{\gamma+\mu_r}.$$

The explicit form of (5) can be written as

$$\begin{cases} s_{k+1} = \dfrac{s_k + \phi_1 \gamma r_k + \phi_1 \mu^*}{1 + \alpha \phi_1 i_k + \phi_1 \mu_s}, \\ i_{k+1} = \dfrac{(1 + \alpha \phi_2 s_{k+1}) i_k}{1 + (\beta + \mu_i)\phi_2}, \\ r_{k+1} = \dfrac{\beta \phi_3 i_{k+1} + r_k}{1 + (\gamma + \mu_r)\phi_3}. \end{cases} \tag{6}$$

PROPOSITION 4.1. *If $s_0 > 0$, $i_0 > 0$ and $r_0 > 0$, then for all stepsize h, the numerical solutions are obtained from (6) are always positive.*

5. Numerical analysis

This section is devoted to numerical interpretation of COVID–19 model using the proposed NSFD scheme simulated with the help of Matlab software. In order to investigate the numerical solutions of the proposed NSFD we consider two cases. At the first simulation, we choose the parameter values $\mu^* = 0.02$, $\mu_s = 0.2$, $\alpha = 0.6$, $\beta = 0.1$, $\gamma = 0.001$ and $\mu_r = \mu_i = 0.02$ with the initial condition $s_0 = 30$, $i_0 = 25$ and $r_0 = 20$ for simulating time 1000 and the stepsize $h = 0.4$. Figure 1 confirms that the NSFD scheme (5) converges to the equilibrium point $E_1 = (0.1, 0, 0)$. In Figure 2, we plot the behaviour of the

NSFD scheme (5) for the parameter values $\beta = 0.1$, $\alpha = 0.05$, $\mu_s = 0.2$, $\mu_i = \mu_r = 0.02$, $\gamma = 0.001$ and $\mu^* = 0.5$ with choosing stepsize $h = 2$ and initial condition $s_0 = 30$, $I_0 = 25$ and $r_0 = 20$. The Figure 2, shows that (s_k, i_k, r_k) approaches to the equilibrium point $E_2 = (2.4, 0.1735, 0.8261)$.

6. Conclusion

In this article, the dynamics of the new COVID–19 model is investigated. The positivity and boundedness of the model is proved. The stability analysis for both equilibrium points is obtained proving that the model is locally asymptotically stable for both equilibrium points. The proposed COVID–19 model is solved using a NSFD scheme. The simulation results show the effective of the NSFD scheme, even for choosing the large stepsize h. As a future research work, we can focus on the fractional–order COVID–19 model and obtain an efficient NSFD scheme which preserves the positivity and stability properties of the fractional order COVID–19 model.

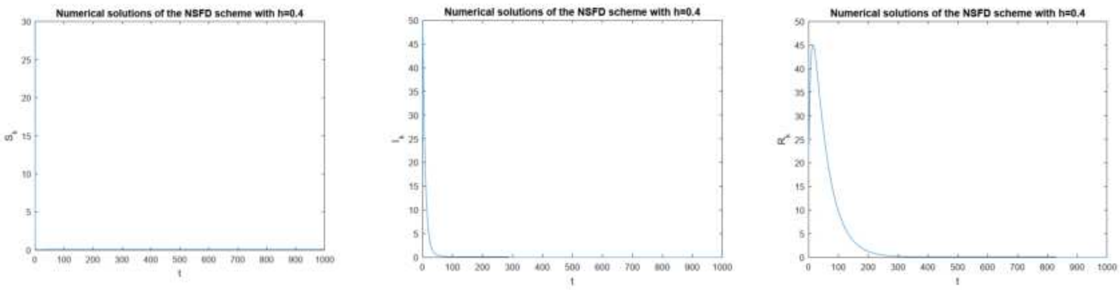

FIGURE 1. Numerical simulation with $h = 0.4$ for the NSFD scheme (5)

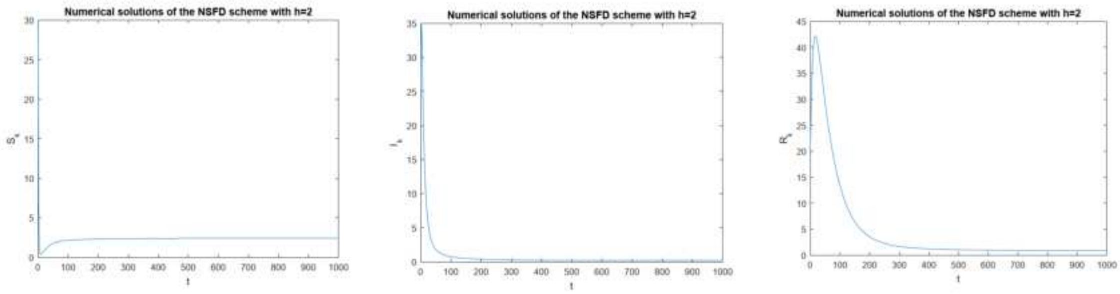

FIGURE 2. Numerical simulation with $h = 2$ for the NSFD scheme (5)

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