Coexistence of Periods in Majority Parallel Dynamical Systems over Directed Graphs

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Abstract. In this talk, we solve the problem of the coexistence of periodic orbits in homogeneous parallel Boolean dynamical systems which are induced by majority function, with a directed dependency graph. In particular, we show that periodic orbits of any period can coexist. This result contrasts with the properties of their counterparts over simple graphs with the same evolution operator, where only fixed points and 2−periodic points can exist and coexist.

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1. Introduction

Many real-world phenomena are modeled as (finite) dynamical systems over large complex networks. For example we can list them as the interactions of gene regulatory networks, the virus spreading thorough a computer network, the spread of a disease through a social network, etc. Such coherent utilizations of (finite) dynamical systems in social network, science and engineering make the research on this topic an interesting and important subject. We devote this paper to study a class of finite dynamical system that we call it majority-PDDS.

Given a finite (non-empty) set of elements X and a function $F : X \to X$, the pair (X, F), or simply F, is named a finite dynamical system. Throughout this work, X is called the state space and F is named the evolution operator of the system.

Let (X, F) be a finite dynamical system, a point $\mathbf{x} \in X$ is called a periodic point of F of period $t > 0$ whenever $F^t(\mathbf{x}) = \mathbf{x}$ and $F^s(\mathbf{x}) \neq \mathbf{x}$ for each $0 < s < t$. We denote by $Per_t(F)$ the set of periodic points of F of period t. In particular, if $t = 1$ then $\mathbf{x}$ is called a fixed point of F, and we denote by $Fix(F)$ the set of fixed points of F. Note that (X, F) is called a fixed point system when all the periodic points are fixed points.

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A *Boolean finite dynamical system* is a finite dynamical system where the state space and the evolution operator are Boolean. More precisely, in a Boolean finite dynamical system (X, F), $X = \{0,1\}^n$ for some natural n and

$$F : \{0,1\}^n \to \{0,1\}^n, \quad F(x_1, \dots, x_n) = (F_1(x_1, \dots, x_n), \dots, F_n(x_1, \dots, x_n))$$

where each F_i is a Boolean function. Corresponding to this system, we consider the underlying graph $G = (V, E)$ on the vertex set $\{1, \dots, n\}$ whose edge/arc set is

$$E = \{(i, j) \ ; \ \text{ the variable } x_i \text{ is involved in the component function } F_j\}.$$

Throughout this work, we assume that for each $1 \leq i \leq n$ the variable x_i appears in the component function F_i, but to simplify, we remove all self-loops of G. The graph G defined in this way is called *dependency graph* of F. The Boolean evolution operator F of a Boolean finite dynamical system can update the (states of) variables in a synchronous or in an asynchronous manner. In the first case, the system is called parallel or synchronous, while in the second case it is named sequential or asynchronous. In the literature, when the dependency graph $G = (V, E)$ is simple (resp. directed), these systems are denoted by PDS and SDS (resp. PDDS and SDDS), respectively. If the evolution operator F is induced by a function

$$f : \{0,1\}^n \to \{0,1\}$$

such that each F_j is computed by the restriction f_j of f to the state of the entry j and the entries i such that $(i, j) \in E$, then the system is called homogeneous. In this setting we simply say F is an $f-$PDS, $f-$SDS (resp. $f-$ PDDS and $f-$SDDS). In this paper, we are going to study $f-$PDDS in the case that f is a majority function.

Following the notations of [3], let

$$sum_n : \{0,1\}^n \to \mathbb{N}, \quad sum_n(a_1, \dots, a_n) = a_1 + \cdots + a_n$$

and assume that the evolution operator F induced by

$$majority_n : \{0,1\}^n \to \{0,1\}, \quad majority_n(a_1, \dots, a_n) = \begin{cases} 1, & sum_n(a_1, \dots, a_n) \geq \lceil \frac{n}{2} \rceil \\ 0, & \text{otherwise} \end{cases}$$

In this situation, we simply say F is a majority-PDS or majority-PDDS depending on G is a simple or directed graph.

Goles and Olivos proved that every periodic point of a majority-PDS has period 2 or 1 (see [2]). Poljak and Turzik showed that for any arbitrary point $\mathbf{x}$, $\mathbf{x}^t$ is a periodic point when $t \geq O(n^2)$ (see [5]). Moreover, Kaaser, Mallmann-Trenn, and Natale ([1]) proved that given an integer t and some graph G, it is NP-hard to decide whether there exists an initial point $\mathbf{x}$ for which $\mathbf{x}^t$ is a periodic point.

Many papers were devoted to the study of majority-PDS while majority-PDDS has net been study well. The main contribution of this paper is to show that periodic orbits of any period can exist and coexist together in majority-PDDS despite the fact that in majority-PDS only fixed points and 2-periodic points can exist and coexist together.

2. Main results

Let F be a majority-PDS over a simple graph G, then, as we mentioned before, $Per_t(F)$ can be a non-empty set only in the case that $t = 1, 2$. As $(1, \dots, 1)$ and $(0, \dots, 0)$ are always fixed points of F, it is clear that $Per_1(F) = Fix(F) \neq \emptyset$. It is worth remarking that in many situations F is a fixed-point system. For example, if G is a tree or a complete graph or a cycle of odd length, then F is a fixed point system, while fixed points and 2-periodic points present simultaneously if G is a cycle of even length (see [4]). In this Section, we show that majority-PDDS behaves completely different from majority-PDS and periodic points of any period may happens for a majority-PDDS.

THEOREM 2.1. *Given* $\{n_1, \ldots, n_r\} \subset \mathbb{N}, r \geq 2$, *there exists a majority-PDDS which presents periodic orbits of periods* $n_1, \ldots, n_r$ *simultaneously.*

PROOF. In order to prove the result, we introduce a specific majority-PDDS F and we find all t that $Per_t(F) \neq \emptyset$. Consider the majority-PDDS F over the directed graph $G = (V, E)$ where

$$V = \{1, \ldots, m, m+1, \ldots, 2m\}$$

and

$$E = \{(i, i+1), (i, i+m+1) \mid 1 \leq i \leq m-1\} \cup \{(i, i+1), (i, i-m+1) \mid m \leq i \leq 2m-1\} \cup \{(2m, 1), (2m, m+1)\}$$

So, $F : \{0,1\}^{2m} \to \{0,1\}^{2m}$ is given by $(F_1, \ldots, F_m, F_{m+1}, \ldots, F_{2m})$ where

$$F_1(x_1, \ldots, x_{2m}) = majority_3(x_1, x_m, x_{2m}),$$

$$F_{m+1}(x_1, \ldots, x_{2m}) = majority_3(x_{m+1}, x_m, x_{2m}),$$

$$\forall 2 \leq i \leq m \;\; F_i(x_1, \ldots, x_{2m}) = majority_3(x_i, x_{i-1}, x_{m+i-1})$$

and

$$\forall m+2 \leq i \leq 2m \;\; F_i(x_1, \ldots, x_{2m}) = majority_3(x_i, x_{i-1}, x_{i-m-1}).$$

Let $\mathbf{a} = (a_1, \ldots, a_m)$ be an arbitrary point of $\{0,1\}^m$ and define

$$\mathbf{b_a} = (a_1, \ldots, a_m, a_1, \ldots, a_m) \in \{0,1\}^{2m}$$

It is straightforward to see that

$$F(\mathbf{b_a}) = (a_m, a_1, \ldots, a_{m-1}, a_m, a_1, \ldots, a_{m-1}).$$

Now consider the finite Boolean dynamical system $R : \{0,1\}^m \to \{0,1\}^m$ where $R(x_1, \ldots, x_m) = (x_m, x_1, \ldots, x_{m-1})$. It is clear that for each positive integer i,

$$F^i(\mathbf{b_a}) = (R^i(\mathbf{a}), R^i(\mathbf{a})).$$

This shows that for each $\mathbf{a} \in \{0,1\}^m$, the orbit of $\mathbf{b_a}$ in F is in one-to-one correspondence with the orbit of $\mathbf{a}$ in R. On the other hand, all orbits of R are periodic orbits and $Per_t(R) \neq \emptyset$ if and only if $t|m$. So, we conclude that for each $\mathbf{a} \in \{0,1\}^m$, the orbit of $\mathbf{b_a}$ in F is a periodic orbit and in particular $Per_t(F) \neq \emptyset$ for each t dividing m.

Now suppose that $\mathbf{b} = (b_1, \ldots, b_m, b_{m+1}, \ldots, b_{2m}) \in \{0,1\}^{2m}$ be such that for each $1 \leq i \leq m$, $b_i \neq b_{m+i}$, then one can easily see that $\mathbf{b}$ is a fixed point of F.

Finally, if $\mathbf{b} = (b_1, \ldots, b_m, b_{m+1}, \ldots, b_{2m}) \in \{0,1\}^{2m}$ is such that $b_i = b_{m+i}$ for some $1 \leq i \leq m$, then one can easily check that there exists a positive integer n and $\mathbf{a} \in \{0,1\}^m$ such that $F^n(\mathbf{b}) = \mathbf{b_a}$ and so the orbit of $\mathbf{b}$ in F converges to the periodic orbit of $\mathbf{b_a}$ in F.

In few words, we have shown that $Per_t(F) \neq \emptyset$ if and only if t divides m. Now to prove the result, for a given $\{n_1, \ldots, n_r\} \subset \mathbb{N}, r \geq 2$, let m be the least common multiple of $n_1, \ldots, n_r$ and F as defined in previous paragraphs. As discussed before $Per_t(F) \neq \emptyset$ for each t dividing m. So, F presents periodic orbits of periods $n_1, \ldots, n_r$ simultaneously and the conclusion follows □

3. Conclusion

Let F be a majority-PDS over a simple graph then, it is well-known that $Per_t(F)$ can be a non-empty set only in the case that $t = 1, 2$. In this paper by a careful study of periodic structure of a specific majority-PDDS, we conclude that for a given $\{n_1, \ldots, n_r\} \subset \mathbb{N}, r \geq 2$, one can find a majority-PDDS which presents periodic orbits of periods $n_1, \ldots, n_r$ simultaneously. This shows that majority-PDDS behave completely different from majority-PDS and studying their periodic structure is more difficult than the case that the dependency graph is a simple graph.

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