سومین کنفرانس سیستم های دینامیکی و نظریه های هندسی
The 3rd Conference on Dynamical Systems & Geometric Theories
Hakim Sabzevari University
Faculty of Mathematics and Computer Science
Iran, Sabzevar
27 & 28 January 2022
دانشگاه حکیم سبزواری
دانشکده ریاضی و علوم کامپیوتر
۷ و ۸ بهمن ۱۴۰۰

n-Jordan $*$-homomorphisms in Fréchet locally C^*-algebras

Shahram Ghaffary[1,*], Javad Jamalzadeh [2]

[1]University of Sistan and Baluchestan, Zahedan, Iran

[2]Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

---

Abstract. Using the fixed point method, we prove the Hyers-Ulam stability and the superstability of n-Jordan $*$-homomorphisms in Fréchet locally C^*-algebras for the following generalized Jensen-type functional equation

$$rf\left(\frac{a+b}{r}\right)+rf\left(\frac{a-b}{r}\right)=2f(a).$$

where r is a fixed real number with $r>1$.

Keywords: n-Jordan $*$-homomorphism, Fréchet locally C^*-algebra, Fréchet algebra, fixed point method, Hyers-Ulam stability

AMS Mathematics Subject Classification [2020]: 17C65, 47H10, 39B52, 39B72, 46L05

---

1. Introduction

The stability of functional equations was first introduced by Ulam in 1940. Hyers gave a partial solution of *Ulam's* problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. Aoki generalized the Hyers' theorem for approximately additive mappings. In 1978, Th.M. Rassias generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. The paper of Th.M. Rassias has provided a lot of influence in the development of what we call Hyers-Ulam-Rassias stability of functional equations.

Theorem 1.1. *Let $f: E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(a+b)-f(a)-f(b)\| \leq \epsilon(\|a\|^p+\|b\|^p) \tag{1}$$

for all $a,b \in E$, where ϵ and p are constants with $\epsilon>0$ and $p<1$. Then there exists a unique additive mapping $T: E \to E'$ such that

$$\|f(a)-T(a)\| \leq \frac{2\epsilon}{2-2^p}\|a\|^p \tag{2}$$

*Speaker. Email address: shahram.ghaffary@pgs.usb.ac.ir,

for all $a \in E$*. If* $p < 0$ *then inequality* (1) *holds for all* $a, b \neq 0$*, and* (2) *holds for* $a \neq 0$*. Also, if the function* $t \to f(ta)$ *from* R *into* E' *is continuous for each fixed* $a \in X$*, then* T *is linear.*

The result of the Th.M. Rassias theorem was generalized by Forti and Gavruta who permitted the Cauchy difference to become arbitrary unbounded. Some results on the stability of functional equations in single variable and nonlinear iterative equations can be found in . G. Isac and Th.M. Rassias were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. The concept of n-Jordan homomorphisms in complex algebras was introduced by Eshaghi Gordji *et al.* J. Jamalzadeh *et al.* introduced the Hyers-Ulam stability and the superstability of n-Jordan $*$-derivations in Fréchet locally C^*-algebras.

During the last decades several stability problems of functional equations have been investigated by many mathematicians (see [1–5]).

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

THEOREM 1.2. *Let* (X, d) *be a complete generalized metric space and let* $J : X \to X$ *be a strictly contractive mapping with Lipschitz constant* $L < 1$*. Then for each given element* $x \in X$*, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n *or there exists a positive integer* n_0 *such that*

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J;
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

In this paper, assume that n is an integer greater than 1.

DEFINITION 1.3. Let A, B be complex algebras. A C-linear mapping $h : A \to B$ is called an n-Jordan homomorphism if

$$h(a^n) = h(a)^n$$

for all $a \in A$.

DEFINITION 1.4. Let A, B be C^*-algebras. An n-Jordan homomorphism $h : A \to B$ is called an n-Jordan $*$-homomorphism if

$$h(a^*) = h(a)^*$$

for all $a \in A$.

DEFINITION 1.5. A topological vector space X is a Fréchet space if it satisfies the following three properties:

(1) it is complete as a uniform space,

(2) it is locally convex,

(3) its topology can be induced by a translation invariant metric, i.e., a metric $d : X \times X \to R$ such that $d(x, y) = d(x + a, y + a)$ for all $a, x, y \in X$.

For more detailed definitions of such terminologies, we can refer to . Note that a ternary algebra is called a ternary Fréchet algebra if it is a Fréchet space with a metric d.

Fréchet algebras, named after Maurice Fréchet, are special topological algebras as follows.

Note that the topology on A can be induced by a translation invariant metric, i.e. a metric $d : X \times X \to R$ such that $d(x, y) = d(x + a, y + a)$ for all $a, x, y \in X$.

Trivially, every Banach algebra is a Fréchet algebra as the norm induces a translation invariant metric and the space is complete with respect to this metric.

A locally C^*-algebra is a complete Hausdorff complex $*$-algebra A whose topology is determined by its continuous C^*-seminorms in the sense that a net $\{a_i\}_{i\in I}$ converges to 0 if and if the net $\{p(a_i)\}_{i\in I}$ converges to 0 for each continuous C^*-seminorm p on A. The set of all continuous C^*-seminorms on A is denoted by $S(A)$. A Fréchet locally C^*-algebra is a locally C^*-algebra whose topology is determined by a countable family of C^*-seminorms. Clearly, any C^*-algebra is a Fréchet locally C^*-algebra.

For given two locally C^*-algebras A and B, a morphism of locally C^*-algebras from A to B is a continuous $*$-morphism φ from A to B. An isomorphism of locally C^*-algebras from A to B is a bijective mapping $\varphi : A \to B$ such that φ and φ^{-1} are morphisms of locally C^*-algebras.

Hilbert modules over locally C^*-algebras are generalization of Hilbert C^*-modules by allowing the inner product to take values in a locally C^*-algebra rather than in a C^*-algebra.

In this paper, using the fixed point method, we prove the Hyers-Ulam stability and the superstability of n-Jordan $*$-homomorphisms in Fréchet locally C^*-algebras for the the following generalized Jensen-type functional equation

$$rf\left(\frac{a+b}{r}\right) + rf\left(\frac{a-b}{r}\right) = 2f(a).$$

2. Stability of n-Jordan $*$-homomorphisms

Lemma 2.1. *Let A, B be linear spaces, and $f : A \to B$ be an additive mapping such that $f(\mu a) = \mu f(a)$ for all $a \in A$ and all $\mu \in T^1 := \{\lambda \in C : |\lambda| = 1\}$. Then the mapping $f : A \to B$ is C-linear.*

Theorem 2.2. *Let A, B be Fréchet locally C^*-algebras, and $f : A \to B$ be a mapping for which there exists a function $\varphi : A \times A \to [0, \infty)$ such that*

$$r\mu f\left(\frac{a+b}{r}\right) + r\mu f\left(\frac{a-b}{r}\right) - 2f(\mu a) \leq \varphi(a, b), \tag{3}$$

$$\|f(a^n) - f(a)^n\| \leq \varphi(a, b), \tag{4}$$

$$\|f(a^*) - f(a)^*\| \leq \varphi(a, b) \tag{5}$$

for all $\mu \in T^1$ and all $a, b \in A$. If there exists an $L < 1$ such that $\varphi(a, b) \leq rL\varphi(\frac{a}{r}, \frac{b}{r})$ for all $a, b \in A$, then there exists a unique n-Jordan $*$-homomorphism $h : A \to B$ such that

$$\|f(a) - h(a)\| \leq \frac{L}{1-L}\varphi(a, 0) \tag{6}$$

for all $a \in A$.

Now, we prove the Hyers-Ulam stability problem for n-Jordan $*$-homomorphisms in Fréchet locally C^*- algebras.

Corollary 2.3. *Let $p \in (0,1)$ and $\theta \in [0,\infty)$ be real numbers. Suppose $f : A \to B$ satisfies*

$$\|r\mu f\left(\frac{a+b}{r}\right) + r\mu f\left(\frac{a-b}{r}\right) - 2f(\mu a)\| \le \theta(\|a\|^p + \|b\|^p),$$

$$\|f(a^n) - f(a)^n\| \le 2\theta\|a\|^p,$$

$$\|f(a^*) - f(a)^*\| \le 2\theta\|a\|^p$$

for all $\mu \in T$ and $a, b \in A$. Then there existes a unique n-Jordan $*$-homomorphism $h : A \to B$ such that

$$\|f(a) - h(a)\| \le \frac{2^p\theta}{2-2^p}$$

for all $a \in A$.

3. Superstability of n-Jordan $*$-homomorphisms

In this section, we prove the superstability of n-Jordan $*$-homomorphisms on Fréchet locally C^*-algebras for the generalized Jensen-type functional equation. we need the following lemma in our main results.

Lemma 3.1. *Let A, B be Fréchet locally C^*-algebras, Let $\theta \ge 0$, p and q be real numbers with $q > 0$ and $p + q \ne 1$. Suppose $f : A \to B$ satisfies $f(0) = 0$ and*

$$\left\|r\mu f\left(\frac{a+b}{r}\right) + r\mu f\left(\frac{a-b}{r}\right) - 2f(\mu a)\right\| \le \theta\|a\|^p\|b\|^q \tag{7}$$

for all $\mu \in T$ and all $a, b \in A$. Then f is C-linear.

Now, we prove the superstability problem for n-Jordan $*$-homomorphisms in Fréchet locally C^*-algebras.

Corollary 3.2. *Let $p, s \in R$ and $\theta, q \in (0,\infty)$ with $p + q \ne 1, s \ne 2$. Let A, B be Fréchet locally C^*-algebras. Suppose $f : A \to B$ satisfies $f(0) = 0$ and*

$$\left\|r\mu f\left(\frac{a+b}{r}\right) + r\mu f\left(\frac{a-b}{r}\right) - 2f(\mu a)\right\| \le \theta\|a\|^p\|b\|^q$$

$$\|f(a^n) - f(a)^n\| \le \theta\|a\|^s$$

for all $\mu \in T$ and all $a, b \in A$. Then f is an n-Jordan $*$-homomorphism.

Corollary 3.3. *Let $p \in R$ and $\theta, q \in (0,1)$ with $p + q \ne 1, 2$. Suppose A, B are Fréchet locally C^*-algebras, $f : A \to B$ satisfies $f(0) = 0$, and*

$$\max\{\|f(a^*) - (f(a))^*\|, \|f(a^n) - (f(a))^n\|, \left\|r\mu f\left(\frac{a+b}{r}\right) + r\mu f\left(\frac{a-b}{r}\right) - 2f(\mu a)\right\|\} \le \theta\|a\|^p\|b\|^q$$

for all $\mu \in T$ and all $a, b \in A$. Then f is an n-Jordan $*$-homomorphism.

4. Conclusion

In this paper, using the fixed point method, we prove the Hyers-Ulam stability and the superstability of n-Jordan $*$-homomorphisms in Fréchet locally C^*-algebras

Acknowledgement

Finally, thank you to the conference organizers.

References

1. A. Inoue, *Locally C^*-algebra*, Mem. Fac. Sci. Kyushu Univ. Ser. A.**25** (1971), 197–235.
2. J. Jamalzadeh, K. Ghasemi, S. Ghaffary, *n-Jordan $*$-derivations in Fréchet locally C^*-algebras*, Int. J. Nonlinear Anal. Appl., **13** (2022) No. 1, 555–562.
3. S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis*, Springer, 2011.
4. C. Park, S. Ghaffary, K. Ghasemi, S.Y. Jang, *Fuzzy n-Jordan *-homomorphisms in induced fuzzy C*-algebras*, Advances in Difference Equations **42** (2012), 1–10.
5. C. Park, S. Ghaffary, K. Ghasemi, *Approximate n-Jordan *-derivations on C*-algebras and JC*-algebras*, Journal of Inequalities and Applications **273** (2012), 1–10.

.