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# A remark on the definition of Ricci flow in Finsler geometry

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Abstract. The Ricci flow in Finsler geometry is defined as a natural generalization of the Hamilton's Ricci flow on Riemannian maniflods. It seems that, this definition is completely suitable for Ricci flow in Finsler geometry. Here, it's stated the reasons for the appropriateness of this definition, similar to the Hamilton's work.

**Keywords:** Ricci flow, Finsler space, gradient flow, Einstein-Hilbert functional

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## 1. Introduction

Ricci flow is a branch of general geometric flows, which is an evolution equation for a Riemannian metric in the set of all Riemannian metrics defined on a manifold. Geometric flow can be used to deform an arbitrary metric into an informative metric, from which one can determine the topology of the underlying manifold and hence innovate numerous progress in the proof of some geometric conjectures. In 1982 Hamilton introduced the notion of Ricci flow on Riemannian manifolds by the evolution equation

$$\frac{\partial}{\partial t}g_{ij} = -2Ric_{ij}, \quad g(t=0) := g_0. \tag{1}$$

The Ricci flow, which evolves a Riemannian metric by its Ricci curvature is a natural analogue of the heat equation for metrics. In Hamilton's celebrated paper [4], it is shown that there is a unique solution to the Ricci flow for an arbitrary smooth Riemannian metric on a closed manifold over a sufficiently short time.

Let $(M, g)$ be a closed Riemannian manifold. One of the most natural functionals one can construct on $\mathcal{M}$ is the so-called Einstein-Hilbert functional $E : \mathcal{M} \longrightarrow \mathbb{R}$, which is the integral of the scalar curvature: $E(g) = \int_M R_g d\mu$, where $R_g$ is the scalar curvature related to $g$. Hamilton has computed the variation of $E$ at $g$, in direction $\frac{\partial}{\partial s}g_{ij} = v_{ij}$, and he has concluded that

$$\frac{d}{ds}E(g) = \int_M < v, \frac{1}{2}Rg - Rc > d\mu.$$

*Speaker. Email address: m.yarahmadi@scu.ac.ir, Note that (twice) the gradient flow of $E$ is

$$\frac{\partial}{\partial s} g_{ij} = 2(\nabla E(g))_{ij} = Rg_{ij} - 2R_{ij}.$$

We note that this equation looks similar to the Ricci flow, but the extra term means that this equation is not parabolic and as such, short time existence is not expected to hold. Dropping the $Rg$ term on the RHS of last formula yields the Ricci flow.
The concept of Ricci flow on Finsler manifolds is defined first by D. Bao, cf., [3], choosing the Ricci tensor introduced by H. Akbar-Zadeh, [2]. It seems to the present authors that, this choice of D. Bao for definition of Ricci tensor, is completely suitable for definition of Ricci flow in Finsler geometry. In fact, in order to define the concept of Ricci tensor, Akbar-Zadeh has used Einstein-Hilbert's functional in general relativity, although it has some computation negligence and introduced definition of Einstein-Finsler spaces as critical points of this functional, similar to the Hamilton's work.

## 2. Main results

The concept of Ricci flow on Finsler manifolds is defined first by D. Bao, cf., [3], choosing the Ricci tensor introduced by H. Akbar-Zadeh, [2]. Let $M$ be a compact Finslerian manifold and $Ric_{jk}$ the symmetric tensor defined by $\frac{1}{2}\frac{\partial^2(F^2\mathcal{R}ic)}{\partial y^j y^k}$. Let $\lambda$ be a differentiable function on $M$. We consider the scalar function on $S(M)$ defined by $\hat{H} = \tilde{H} - \lambda(x)\mathcal{R}ic$, where $\tilde{H} = g^{jk}Ric_{jk}$. Akbar-Zadeh consider the functional $E(g) = \int_{S(M)} \hat{H} d\mu$ as energy-functional similar to Hamilton's approach in Finsler space, cf. [1]. Here, we look for the gradient flow of the energy functional introduced by Akbar-Zadeh. It is computed that the variation of $E$ at $g$, in direction $\frac{\partial}{\partial s} g_{ij} = v_{ij}$ is

$$\frac{d}{ds}E(g) = -< A, v >= -\int_{S(M)} A^{jk} v_{jk}\, d\mu,$$

where $<,>$ denotes the global scalar product and $A$ is defined by

$$A^{jk} = Ric^{jk} - \lambda\mathcal{R}ic l^j l^k - (n\tau - \phi)l^j l^k - \hat{H}(g^{jk} - \frac{1}{2}nu^j u^k).$$

The (twin) gradient flow of energy functional introduced by Akbar-Zadeh is $\frac{\partial}{\partial s} g_{jk} = -2A_{jk}$, that is

$$\frac{\partial}{\partial s} g_{jk} = -2(Ric_{jk} - \lambda\mathcal{R}ic l_j l_k - (n\tau - \phi)l_j l_k - \hat{H}(g_{jk} - \frac{1}{2}nl_j l_k)).$$

By multiplying the two sides by $l^j$ and $l^k$ successively we get

$$\frac{d}{ds}\ln(F(t)) = -(1 - \frac{1}{2}n\lambda)Ric + (n\tau - \phi) - (1 - \frac{1}{2}n)\tilde{H}.$$

Case $\lambda = 0$. In this case the integral $E(g)$ (the energy functional introduced by Akbar-Zadeh) is reduced to $E_1(g) = \int_{SM} \tilde{H} d\mu$. The derivative of $E_1(g)$ is $E_1'(g) = -< \tilde{A}, v >$, where

$$\tilde{A}_{ij} = Ric_{ij} - n\tau l_i l_j - \tilde{H}(g_{ij} - \frac{1}{2}nl_i l_j).$$

The (twin) gradient flow of energy functional $E_1(g)$ is

$$\frac{\partial}{\partial s} g_{ij} = -2\tilde{A}_{ij}.$$

By multiplying the two sides by $u^j$ and $u^k$ successively we get

$$\frac{d}{ds}\ln(F(t)) = -H(u,u) + n\tau - (1-\frac{1}{2}n)\tilde{H}.$$

Similar to Hamilton's approach for definition of Ricci flow, we note that this equation looks similar to the Ricci flow, but the extra term means that this equation is not parabolic and as such, short time existence is not expected to hold. Dropping the $n\tau - (1-\frac{1}{2}n)\tilde{H}$ term on the RHS of last formula yields the Finsler Ricci flow $\frac{d}{ds}\ln(F(t)) = -H(u,u)$. Therefore the Finsler Ricci flow introduced by David Bao is completely suitable for definition of Ricci flow in Finsler geometry.

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