Geometric Analysis of the Lie Algebra of Killing Vector Fields for a Significant Cosmological Model of Rotating Fluids

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Abstract. The investigation of rotating fluids in the context of general relativity received remarkable consideration principally after Godel proposed relativistic model of a rotating dust universe. In this paper, a comprehensive analysis regarding the structure of the Lie algebra of Killing vector fields for a specific solution of field equations describing the behaviour of rotating fluid models is presented. Killing vector fields can be undoubtedly reckoned as one of the most substantial types of symmetries and are denoted by the smooth vector fields which preserve the metric tensor. Additionally, the flow corresponding to a Killing vector field generates a symmetry in a way that if each point moves on an object at the same distance in the direction of the Killing vector field then distances on the object will not distorted at all. Therefore, Killing vector fields are inherently expected to be of significant application in the study of geodesic motion. When one investigates the Lagrangian explaining the motion of a particle, one can realize that Killing vectors are the symmetries of the system and lead to conserved canonical momenta analogous to cyclic coordinates in classical mechanics. Taking into account the outstanding properties declared above, in this paper, we specifically concentrate on detailed investigation of the Killing vector fields by reexpressing the analyzed cosmological solution in the orthogonal frame. Significantly, for the resulted Lie algebra of Killing vector fields, the associated basis for the original Lie algebra is determined in which the Lie algebra will be appropriately decomposed into an internal direct sum of subalgebras, where each summand is indecomposable.

Keywords: Killing vector fields, five dimensional spacetime, rotating fluids

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1. Introduction

Killing vector fields can be regarded as one of the most significant types of symmetries and are considered as the smooth vector fields which preserve the metric tensor. These

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vector fields are extensively applied in various physical fields including in classical mechanics and are closely related to conservation laws. Specifically, remarkable applications of Killing vector fields in relativistic theories is undeniable. The noticeable fact is that the flow corresponding to a Killing vector field generates a symmetry in a way that if each point moves on an object at the same distance in the direction of the Killing vector field then distances on the object will not distorted at all. In particular, a vector field K is a Killing field if the Lie derivative with respect to K of the metric g vanishes. Moreover, the Lie bracket of two Killing vector fields is still a Killing field and the Killing fields on a manifold M thus form a Lie subalgebra of vector fields on M which can be considered as the isometry group of the manifold whenever M is complete [2,3]. Taking into account the significant properties declared above, one naturally expects Killing vectors to be of substantial use in the study of geodesic motion. When one investigates the Lagrangian explaining the motion of a particle, one can realize that Killing vectors are the symmetries of the system and lead to conserved canonical momenta analogous to cyclic coordinates in classical mechanics. Furthermore, One can also try to obtain another conserved quantity related to the spacetime itself, if the background metric contains globally well defined Killing vectors [1,5].
Considering the fact that stationary Kaluza-Klein perfect fluid models in standard Einstein theory are not available in literature, obtaining and analyzing such solutions is so constructive in order to investigate the effects of dimensionality on the different physical parameters. In [4], R. Tikekar and L. K. Patel have formulated the Kaluza-Klein field equations for cylindrically symmetric rotating distributions of perfect fluid. They have reported a set of physically viable solutions which is believed to be the first such Kaluza-Klein solutions and it includes the Kaluza-Klein counterpart of Davidson's solution.
In the following, according to [4], we will present a brief description of Kaluza-Klein field equations for stationary cylindrically symmetric fluid models in standard Einstein theory. For further complete information refer to [4].
A general stationary cylindrically symmetric five dimensional spacetime is denoted by the following metric:

$$ds^2 = D^2(dt + Hd\phi)^2 - A^2dr^2 - B^2dz^2 - r^2C^2d\phi^2 - E^2d\psi^2, \tag{1}$$

where t is the time coordinate, r, z and ϕ are cylindrical polar coordinates, ψ represents the coordinate corresponding to the extra spatial dimension and A, B, C, D and H are functions of the radial coordinate r only. By expressing with respect to pentad

$$\theta^1 = Adr, \quad \theta^2 = Bdz, \quad \theta^3 = rCd\phi, \quad \theta^4 = Ed\psi, \qquad \theta^5 = D(dt + Hd\phi), \tag{2}$$

the metric (1) has the following form:

$$ds^2 = (\theta^5)^2 - (\theta^1)^2 - (\theta^2)^2 - (\theta^3)^2 - (\theta^4)^2. \tag{3}$$

If the metric (1) is to denote the spacetime of a stationary perfect fluid rotating about the regular axis $r = 0$, the metric coefficients will be related to the dynamical variables through the Einstein field equations which are in the pentad notation applying the system of units rendering $c = G = 1$, adopted in the form

$$\begin{cases} \mathbf{R}_{(11)} = \mathbf{R}_{(22)} = \mathbf{R}_{(33)} = \mathbf{R}_{(44)} = -\dfrac{8\pi}{3}(\rho - p), \\ \mathbf{R}_{(55)} = -\dfrac{16\pi}{3}(\rho + 2p), \\ \mathbf{R}_{(35)} = 0, \end{cases} \tag{4}$$

The field equations comprise a system of six equations relating the two physical parameters ρ and p of the fluid and the six metric coefficients A, B, C, D, E and H. Accordingly, the equation $\mathbf{R}_{(35)} = 0$ in (4) yields the following two significant identities:

$$H = \alpha r^2, \tag{5}$$

$$a + c = 2b + 3d, \tag{6}$$

where α is the arbitrary constant of integration. In [4] certain specific cases for physical relevance which follow for certain particular choices of the free parameters, are discussed. In this paper, we will comprehensively analyze the structure of the Lie algebra of Killing vector fields for the following three specific solutions which is reported in [4].
When $a = -1/2$, $b = e = c = -d = 1/4$, $\alpha^2 = k^2$, the Kaluza-Klein equations are all satisfied and the spacetime of this class of solutions has the metric

$$\begin{aligned} ds^2 &= (1 + k^2r^2)^{-1/2}(dt + kr^2d\phi)^2 - (1 + k^2r^2)^{-1}dr^2 \\ &-(1 + k^2r^2)^{1/2}\big(dz^2 + r^2d\phi^2 + d\psi^2\big). \end{aligned} \tag{7}$$

which denotes a five dimensional spacetime of a cylindrically symmetric stationary fluid with constant density and pressure related by this equation of state: $\rho + p = 0$. By setting $\Lambda = -(3/2)k^2$, the metric above denotes a five dimensional solutions of the field equations: $\mathbf{R}_{ij} = \Lambda g_{ij}$, where Λ represents the cosmological constant.

This paper is organized as follows: In the next section, we have specifically concentrated on complete investigation of the problem of Killing vector fields for our analyzed physically viable five dimensional cosmological solution. First of all, by considering the Lagrangian which is determined directly from the metric, we will compute the corresponding symmetries which preserve the metric tensor. Additionally, the flow corresponding to a Killing vector field generates a symmetry in a way that if each point moves on an object at the same distance in the direction of the Killing vector field then distances on the object will not distorted at all. Therefore, Killing vector fields are inherently expected to be of significant application in the study of geodesic motion. When one investigates the Lagrangian explaining the motion of a particle, one can realize that Killing vectors are the symmetries of the system and lead to conserved canonical momenta analogous to cyclic coordinates in classical mechanics. Taking into account the outstanding properties declared above, the next section of this paper is particularly devoted to detailed investigation of the Killing vector fields by reexpressing the metric (7) in the orthogonal frame. Meanwhile, some concluding remarks are declared at the end of the paper.

2. Classification of Killing Vector Fields

Let (M, g) be an arbitrary Lorentzian manifold and $\Im$ be a smooth vector field on M. A curve $\gamma : \mathbb{R} \longrightarrow M$ whose tangent vector at every point $p \in \gamma$ is equal to $\Im$ is denoted by an integral curve of $\Im$. Significantly, for a given congruence a one-parameter family of diffeomorphisms from M onto itself can be associated which is described as follows: corresponding to each $s \in \mathbb{R}$, designate a map $\mathcal{F}_s : M \longrightarrow M$, where $\mathcal{F}_s(p)$ is the point parameter distance s from p along $\Im$, i.e. if $p = \gamma(\wp_0)$ then $\mathcal{F}_s(p) = \gamma(\wp_0 + s)$. Furthermore, from the algebraic point of view, considering the composition law $\mathcal{F}_s o \mathcal{F}_t = \mathcal{F}_{s+t}$, the identity $\mathcal{F}_0$ and the inverse $(\mathcal{F}_s)^{-1} = \mathcal{F}_{-s}$, these transformations construct an abelian group. Specifically to a metric tensor g on M the Lie derivative is defined by:

$$(\mathcal{L}_{\Im} g)_p = \lim_{\delta\wp \longrightarrow 0} \frac{g_p - (\mathcal{F}_{\delta\wp})^* g_{\mathcal{F}_{\delta\wp}}(p)}{\delta\wp} \tag{8}$$

The remarkable fact is that the Lie derivative of g entails the pull-back $\mathcal{F}_s^*$ which maps a covector at $\mathcal{F}_s(p)$ to a covector at p, mainly due to the fact that the components of g transform covariantly. It can be demonstrated that:

$$(\mathcal{L}_{\Im} g)_{\mu\nu} = \nabla_\mu \Im_\nu + \nabla_\nu \Im_\mu. \tag{9}$$

Meanwhile, if the metric does not change under the transformation $\mathcal{F}_s$, the transformation is called an isometry and that the metric possesses a symmetry. In this case $\mathcal{L}_{\Im} g = 0$, which leads to the following identity [3, 5]:

$$\nabla_\mu \Im_\nu + \nabla_\nu \Im_\mu = 0 \tag{10}$$

This relation is denoted by Killing's equation and a vector $\Im$ which satisfies (10) is called a Killing vector. It is noticeable that this identity contains the metric implicity which is hidden in ∇. In addition, the symmetries of a spacetime explicitly leads to determining the vectors which satisfy the Killing equation; this can be thoroughly fulfilled either by inspection or via integrating (10). An isometry is a distance preserving mapping among different spaces.
In this section, we apply an orthogonal frame to obtain the Killing vector fields for the metric (7). First of all, we set up a five dimensional spacetime with coordinates $[t, r, z, \phi, \psi]$ denoted by ϖ given by:

$$\begin{aligned} \varpi = \Bigg[& \frac{1}{\sqrt{k^2r^2+1}}\, dr, \left(k^2r^2+1\right)^{1/4} dz, \ \left(k^2r^2+1\right)^{1/4} dx, \left(k^2r^2+1\right)^{1/4} dy, \\ & \frac{1}{\left(k^2r^2+1\right)^{1/4}}\, dt + \frac{kr^2}{\left(k^2r^2+1\right)^{1/4}}\, dx \Bigg] \end{aligned} \tag{11}$$

Then we define a coframe and calculate the structure equations for this coframe as follows:

$$\begin{cases} d\Theta_1 = 0, \quad d\Theta_2 = \dfrac{k^2 r}{2\sqrt{k^2r^2+1}}\, \Theta_1 \wedge \Theta_2, \quad d\Theta_3 = \dfrac{3k^2r^2+2}{2r\sqrt{k^2r^2+1}}\, \Theta_1 \wedge \Theta_3, \\ d\Theta_4 = \dfrac{k^2 r}{2\sqrt{k^2r^2+1}}\, \Theta_1 \wedge \Theta_4, \quad d\Theta_5 = 2k\ \Theta_1 \wedge \Theta_3 - \dfrac{k^2 r}{2\sqrt{k^2r^2+1}}\, \Theta_1 \wedge \Theta_5. \end{cases} \tag{12}$$

Therefore, we can state the following theorem:

Theorem 2.1. *Taking into account the pentad (2), the metric (7) is expressed by (3) in the orthogonal frame. Subsequently, the following seven Killing vectors corresponding* *to the metric (7) are resulted in the adapted frame:*

$$\begin{cases}
(\mathbf{1}): \mathbf{K_1} = t\left(k^2r^2+1\right)^{1/4}\ \mathbf{E_2} + \left(k^2r^2+1\right)^{1/4}\ kzr\ \mathbf{E_3} + z\left(k^2r^2+1\right)^{3/4}\ \mathbf{E_5}, \\
(\mathbf{2}): \mathbf{K_2} = \left(k^2r^2+1\right)^{1/4}\ kyr\ \mathbf{E_3} + \left(k^2r^2+1\right)^{1/4}\ t\ \mathbf{E_4} + y\left(k^2r^2+1\right)^{3/4}\ \mathbf{E_5}, \\
(\mathbf{3}): \mathbf{K_3} = \dfrac{1}{\left(k^2r^2+1\right)^{1/4}}\ \mathbf{E_5}, \\
(\mathbf{4}): \mathbf{K_4} = \dfrac{\left(k^2r^2+1\right)^{1/4}\ r}{k}\ \mathbf{E_3} + \dfrac{r^2}{\left(k^2r^2+1\right)^{1/4}}\ \mathbf{E_5}, \\
(\mathbf{5}): \mathbf{K_5} = \left(k^2r^2+1\right)^{1/4}\ y\ \mathbf{E_2} - \left(k^2r^2+1\right)^{1/4}\ z\ \mathbf{E_4}, \\
(\mathbf{6}): \mathbf{K_6} = -\left(k^2r^2+1\right)^{1/4}\ \mathbf{E_4}, \\
(\mathbf{7}): \mathbf{K_7} = -\left(k^2r^2+1\right)^{1/4}\ \mathbf{E_2}.
\end{cases}$$

Furthermore, here are the structure equations for the Lie algebra of Killing vectors denoted by $\mathcal{K}$:

$$[\mathbf{K_1},\mathbf{K_2}] = -\mathbf{K_5},\quad [\mathbf{K_1},\mathbf{K_3}] = \mathbf{K_7},\quad [\mathbf{K_1},\mathbf{K_5}] = -\mathbf{K_2},,\quad [\mathbf{K_1},\mathbf{K_7}] = k^2\ \mathbf{K_4}+\mathbf{K_3},$$

$$[\mathbf{K_2},\mathbf{K_5}] = \mathbf{K_1},\quad [\mathbf{K_2},\mathbf{K_6}] = k^2\ \mathbf{K_4}+\mathbf{K_3},\quad [\mathbf{K_5},\mathbf{K_6}] = -\mathbf{K_7},\quad [\mathbf{K_5},\mathbf{K_7}] = \mathbf{K_6}.$$

Significantly, due to above theorem we have:

Corollary 2.2. *By considering the following basis for the original Lie algebra of Killing vector fields $\mathcal{K}$, it will decompose into an internal direct sum of subalgebras, where each summand is indecomposable.*

$$\left\{\mathbf{F_1},\mathbf{F_2},\mathbf{F_3},\mathbf{F_4},\mathbf{F_5},\mathbf{F_6},\mathbf{F_7}\right\} := \left\{\mathbf{K_1},\mathbf{K_2},\mathbf{K_3}+k^2\ \mathbf{K_4},\mathbf{K_5},\mathbf{K_6},\mathbf{K_7},\mathbf{K_4}\right\}$$

The expression of $\mathcal{K}$ in this new basis described above, will be denoted by $\tilde{\mathcal{K}}$. Meanwhile, $\mathcal{A}$ is a matrix which defines a Lie algebra isomorphism from $\mathcal{K}$ to $\tilde{\mathcal{K}}$ (the Lie algebra defined by the direct sum of indecomposable Lie subalgebras) given by:

$$\mathcal{A} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -k^2 & 1 & 0 & 0 & 0
\end{pmatrix}$$

The commutator table of $\tilde{\mathcal{K}}$ is illustrated in Table 8, where the entry in the i^{th} row and j^{th} column is defined as $[F_i, F_j] = F_iF_j - F_jF_i, \quad i, j = 1, ..., 7$.

Table 1. Commutation relations satisfied by infinitesimal generators for the Lie algebra $\tilde{\mathcal{K}}$

[,]	$\mathbf{F_1}$	$\mathbf{F_2}$	$\mathbf{F_3}$	$\mathbf{F_4}$	$\mathbf{F_5}$	$\mathbf{F_6}$	$\mathbf{F_7}$
$\mathbf{F_1}$	0	$-\mathbf{F_4}$	$\mathbf{F_6}$	$-\mathbf{F_2}$	0	$\mathbf{F_3}$	0
$\mathbf{F_2}$	$\mathbf{F_4}$	0	$\mathbf{F_5}$	$\mathbf{F_1}$	$\mathbf{F_3}$	0	0
$\mathbf{F_3}$	$-\mathbf{F_6}$	$-\mathbf{F_5}$	0	0	0	0	0
$\mathbf{F_4}$	$\mathbf{F_2}$	$-\mathbf{F_1}$	0	0	$-\mathbf{F_6}$	$\mathbf{F_5}$	0
$\mathbf{F_5}$	0	$-\mathbf{F_3}$	0	$\mathbf{F_6}$	0	0	0
$\mathbf{F_6}$	$-\mathbf{F_3}$	0	0	$-\mathbf{F_5}$	0	0	0
$\mathbf{F_7}$	0	0	0	0	0	0	0

3. Conclusion

The general theory of relativity which can be regarded as the field theory of gravitation is fundamentally governed by the Einstein field equations (EFE). These equations are extremely nonlinear and are demonstrated in terms of the Lorentzian metric g_{ab}. Taking into account this nonlinearity, obtaining their exact solutions is totally difficult. Hence, it has been one of the basic problems in general relativity to analyze the solutions of the Einstein field equations by means of the symmetries they possess. In the current research, we have exhaustively analyzed the structure of the Lie algebra of Killing vector fields for a specific cosmological solution in standard Einstein theory. This physically remarkable five dimensional spacetime, describes the total behaviour of rotating fluids. Particularly, by reexpressing the analyzed metric in the orthogonal coframe, the corresponding Killing vector fields which can be regarded as one of the most significant types of symmetries and are considered as the smooth vector fields which preserve the metric tensor are thoroughly calculated. Significantly, for the resulted Lie algebra of Killing vector fields, the associated basis for the original Lie algebra is determined in which the Lie algebras will be appropriately decomposed into an internal direct sum of subalgebras, where each summand is indecomposable.

References

1. G. Bluman, A.F. Cheviakov, S. Anco, Application of symmetry methods to partial differential equations, Springer, (2010).
2. S. M. Carroll, Spacetime and Geometry: An Introduction to General Relativity, Addison-Wesley, (2004).
3. G. Hall, Symmetries and Curvature Structure in General Relativity (World Scientific Lecture Notes in Physics), Singapore: World Scientific Pub. Co., (2004).
4. R. Tikekar, L.K. Patel, *Rotating cylindrically symmetric Kaluza-Klein fluid model*, Pramana Journal of Physics., 55 (2000), 361–368.
5. M. Tsamparlis, A. Paliathanasis, *Lie and Noether symmetries of geodesic equations and collineations*, General Relativity and Gravitation., 42.12 (2010), 2957–2980.