سومین کنفرانس سیستم های دینامیکی و نظریه های هندسی
The 3rd Conference on Dynamical Systems & Geometric Theories
Hakim Sabzevari University
Faculty of Mathematics and Computer Science
Iran, Sabzevar
27 & 28 January 2022
دانشگاه حکیم سبزواری
دانشکده ریاضی و علوم کامپیوتر
۷ و ۸ بهمن ۱۴۰۰

The shadowing and ergodic shadowing properties of semigroup actions on non-compact metric spaces

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Abstract. In this talk, we introduce the notions of shadowing and ergodic shadowing properties of finitely generated semigroup actions on non-compact metric spaces which are dynamical properties and equivalent to the classical definitions in case of compact metric spaces.

Keywords: ergodic shadowing property, shadowing property, semigroup actions

AMS Mathematics Subject Classification [2020]: 37C50, 37B05

1. Introduction and Preliminaries

The concept of shadowing was emanated from the Anosov closing lemma and because of its rich consequences, it has a significant role in the study of dynamical systems. It is considerably developed in recent years and many authors have studied several kinds of shadowing including ergodic shadowing [2], $\underline{d}$-shadowing, and average shadowing, which have the common motivation of studying the behavior of a dynamical system by using the closeness of approximate orbits and true orbit. The shadowing and average shadowing properties of IFSs(iterated function systems) were introduced by Bahabadi in [1]. He obtained average shadowing property of an iterated function system implies chain transitivity. Ergodic shadowing property of semigroup actions with finitely many generators were introduced in [4], and the author showed that if a semigroup G has the shadowing property then the ergodic shaowing property is equivalent to some kind of specification which is called pseudo-orbital specification.

However, the definitions of shadowing and ergodic shadowing properties for a continuous map f and also for the finitely generated semigroup actions on a compact metric space depend on the metrics on non-compact metric spaces. In other words, the map f (or semigroup G) has the shadowing (ergodic shadowing) property with respect to one metric, may does not have the shadowing (ergodic shadowing) property with respect to

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another metric inducing the same topology (see [3, Example 2.2] and Example 1.2). Lee et al. [3] introduced the notions of ε-chain and shadowing property for homeomorphisms on non-compact metric spaces, which are dynamical properties and equivalent to the classical definitions in case of compact metric spaces. Here, we extend the notions of shadowing and ergodic shadowing properties to case of finitely generated semigroup actions on non-compact metric spaces and we show that the definitions of shadowing in the new notions are dynamical properties.

Definition 1.1. [5] Let X and Y be two metric spaces. We say that two semigroups F and G with generating sets $\{id, f_1, \ldots, f_m\}$ and $\{id, g_1, \ldots, g_m\}$ on X and Y, receptively, are *(topologically) conjugate* if there is a homeomorphism $h : X \to Y$ such that $h \circ f_i = g_i \circ h$ for all $i = 1, \ldots, m$. The homeomorphism h is called a *conjugacy* between F and G.

A property P is called a *dynamical property* if a semigroup G has the property P, then any other semigroup F which is conjugate to G also has the property P. Note that shadowing and ergodic shadowing properties of semigroup action on compact metric spaces are independent of metric and they are dynamical properties. However, they depend on the metrics on non-compact metric spaces, as we see in the following example.

Example 1.2. Let $T : \mathbb{R} \to \mathbb{S}^1 \setminus \{(0,1)\}$ be a map given by

$$T(t) = \left(\frac{2t}{1+t^2}, \frac{t^2-1}{t^2+1} \right), \quad \text{for all } t \in \mathbb{R},$$

and let $X = T(\mathbb{Z})$. Let d be the metric on X induced by the Riemannian metric on S^1, and let d' be a discrete metric on X. It is clear that d and d' induce the same topology on X. Let $g_1 : X \to X$ be a homeomorphism defined by $g_1(a_i) = a_{i+1}$. Denote by G the finitely generated semigroup action associated with $\{id, g_1, g_2\}$, where g_2 is any homeomorphism on X. Since the metric d' is discrete, it is easy to see that G has the ergodic shadowing property with respect to d'. We show that g_1 does not have the shadowing property with respect to d. Therefore the semigroup G does not have the shadowing property. By contradiction, let $g_1 : X \to X$ have the shadowing property. For $\varepsilon = \frac{1}{4}$, let $\delta > 0$ be an ε-modulus of the shadowing property of the mapping g_1. Choose $N_0 \in \mathbb{N}$ such that $d'(a_{N_0}, a_{-N_0}) < \frac{\delta}{2}$. For any $i \geq 0$, put $j := i \bmod 2N_0$. Then, the sequence $\{x_i\}_{i \geq 0}$ given by

$$x_i = \begin{cases} a_j, & j \in \{0, 1, 2, \ldots, N_0 - 1\}, \\ a_{j-2N_0}, & j \in \{N_0, N_0 + 1, \ldots, 2N_0 - 1\}. \end{cases}$$

is a δ-pseudo orbit for g_1. So, there is a point $z \in X$ such that $d(g_1^i(z), x_i) < \varepsilon$, for any $i \geq 0$. Since for any $z \in X$, $g_1^i(z)$ attract to $(0,1)$, so we can find an integer $i \in \mathbb{N}$ such that $d(g_1^i(z), x_i) \geq \varepsilon$, which is a contradiction. So, G does not have the shadowing and ergodic shadowing properties with respect to d.

2. The shadowing and ergodic shadowing on non-compact metric spaces

In this section, we define the notions of shadowing and ergodic shadowing properties for the finitely generated semigroup actions on non-compact metric spaces, which are independent of metrics.

Let $\mathcal{C}(X)$ be the collection of all continuous functions from X to $(0, \infty)$. Let X be a metrizable space and let G be a finitely generated semigroup action with the set of generators $\{id, g_1, \ldots, g_m\}$. For a sequence $\xi = \{x_i\}_{i\geq 0} \subset X$, $\delta \in \mathcal{C}(X)$, and $\omega = \omega_0\omega_1\omega_2 \ldots \in \Sigma^m$, put

$$Np(\xi, G, \omega, \delta) = \{i \in \mathbb{Z}^+ : d(g_{\omega_i}(x_i), x_{i+1}) \geq \delta(g_{\omega_i}(x_i))\},$$
$$Np^c(\xi, G, \omega, \delta) = \mathbb{Z}^+ \setminus Np(\xi, G, \omega, \delta),$$

and

$$Np_n(\xi, G, \omega, \delta) = Np(\xi, G, \omega, \delta) \cap \{0, \ldots, n-1\}.$$

Given a sequence $\xi = \{x_i\}_{i\geq 0}$ and a point $z \in X$, consider

$$Ns(\xi, G, \omega, z, \delta) = \{i \in \mathbb{Z}^+ : d(g^i_\omega(z), x_i) \geq \delta(g^i_\omega(z))\},$$
$$Ns^c(\xi, G, \omega, z, \delta) = \mathbb{Z}^+ \setminus Ns(\xi, G, \omega, z, \delta),$$

and

$$Ns_n(\xi, G, \omega, z, \delta) = Ns(\xi, G, \omega, z, \delta) \cap \{0, \ldots, n-1\}.$$

Definition 2.1. Let X be a metrizable space, let G be a finitely generated semigroup action with the set of generators $\{id, g_1, \ldots, g_m\}$, and let $\delta \in \mathcal{C}(X)$.

(1) [5] For $w \in \mathcal{A}^m$ and $x, y \in X$, a (δ, w)-chain of semigroup G from x to y is a finite sequence $x_0 = x, x_1, \ldots, x_n = y$ such that $d(g_{w_i}(x_i), x_{i+1}) < \varepsilon(g_{w_i}(x_i))$, for all $i = 1, \ldots, n-1$.

(2) We say that $\{x_i\}_{i\geq 0} \subset X$ is a (δ, ω)-pseudo orbit of G for some $\omega = \omega_0\omega_1 \ldots \in \Sigma^m$, if for any $i \in \mathbb{Z}^+$, $d(g_{\omega_i}(x_i), x_{i+1}) < \delta(g_{\omega_i}(x_i))$.

(3) We say that $\{x_i\}_{i\geq 0} \subset X$ is a (δ, ω)-*ergodic pseudo orbit* of G for some $\omega = \omega_0\omega_1 \ldots \in \Sigma^m$ provided that the set $Np(\xi, G, \omega, \delta)$ has zero density, that is,

$$\lim_{n\to\infty} \frac{|Np_n(\xi, G, \omega, \delta)|}{n} = 0.$$

Definition 2.2. Let X be a metrizable space, let G be a finitely generated semigroup action with the set of generators $\{id, g_1, \ldots, g_m\}$. We say that

(1) G has the shadowing property, if for every $\epsilon \in \mathcal{C}(X)$, there is $\delta \in \mathcal{C}(X)$ such that for every (δ, ω)-pseudo orbit $\{x_i\}_{i\geq 0}$ of G, for some $\omega \in \Sigma^m$, there is a point $z \in X$ satisfying $d(g^i_\omega(x), x_i) < \epsilon(g^i_\omega(x))$ for all $i \geq 0$.

(2) G has the ergodic shadowing property if for each $\epsilon \in \mathcal{C}(X)$, there exists $\delta \in \mathcal{C}(X)$ such that every (δ, ω)-ergodic pseudo orbit ξ of G can be ϵ-ergodic shadowed by some point z in X, that is, there exists $\varphi \in \Sigma^m$ with $\varphi_i = \omega_i$ for $i \in Np^c(\{x_i\}_{i\geq 0}, G, \omega, \delta)$, such that

$$\lim_{n\to\infty} \frac{|Ns_n(\{x_i\}_{i\geq 0}, G, \varphi, z, \epsilon)|}{n} = 0.$$

In the following, we show that Definition 2.2 for the semigroup G on the non-compact metric space X can be preserved by conjugacy. Hence, they do not depend on the choices of metrics on X. For this, we need two lemmas.

Lemma 2.3. [3, Lemmas 2.7 and 2.8] *Let (X, d) and (Y, d') be two metric spaces.*

(1) *A function f from X to Y is continuous if and only if, for any $\varepsilon \in \mathcal{C}(Y)$, there exists $\delta \in \mathcal{C}(X)$ such that if $d(x, y) < \delta(x)$ $(x, y \in X)$, then $d'(f(x), f(y)) < \varepsilon(f(x))$.*

(2) *For every $\alpha \in \mathcal{C}(X)$, there exists $\gamma \in \mathcal{C}(X)$ such that*

$$\gamma(x) \leq \inf \{\alpha(z) : z \in B(x, \gamma(x))\}. \tag{1}$$

The next lemma is cited from [5] and is an immediate result of Lemma 2.3.

LEMMA 2.4. *[5] Let (X,d) be a metric space and let $f_i : X \to X$ $(i = 1,\dots,m)$ be continuous maps. Then, for every $\varepsilon \in \mathcal{C}(X)$, there exists $\delta \in \mathcal{C}(X)$ such that if $d(x,y) < \delta(x)$, then $d(f_i(x), f_i(y)) < \varepsilon(f_i(x))$ for all $i = 1,\dots,m$.*

PROPOSITION 2.5. *Let X be a metric space and let G be a semigroup action with generating set $\{id, g_1,\dots,g_m\}$. Then the shadowing and ergodic shadowing properties of G introduced in Definition 2.2, are dynamical properties.*

PROOF. Let G and F be two semigroups generated by $G_1 = \{id, g_1,\dots,g_m\}$ and $F_1 = \{id, f_1,\dots,f_m\}$ on the metric space (X,d) and (Y,d'), respectively. Suppose that G and F are topologically conjugate with conjugacy $h : X \to Y$. We show that the ergodic shadowing property preserves by topological conjugacy. Assume that G has the ergodic shadowing property. For every $\varepsilon' \in \mathcal{C}(Y)$, there exists $\varepsilon \in \mathcal{C}(X)$ such that if $d(x,y) < \varepsilon(x)$, then $d'(h(x), h(y)) < \varepsilon'(h(x))$. Take $\delta \in \mathcal{C}(X)$ as an ε-modulus of ergodic shadowing property of G, and let $\delta' \in \mathcal{C}(Y)$ be such that if $d'(x,y) < \delta'(x)$, then $d(h^{-1}(x), h^{-1}(y)) < \delta(h^{-1}(x))$. Let $\{x_i\}_{i\geq 0} \subseteq Y$ be a (δ',ω)-ergodic pseudo orbit of F for $\omega = \omega_0\omega_1\ldots \in \Sigma^m$.
We show that $\{h^{-1}(x_i)\}_{i\geq 0}$ is a (δ,ω)-ergodic pseudo orbit of G. Indeed for any $i \in Np^c(\{x_i\}_{i\geq 0}, F, \omega, \delta')$, we have $d'(f_{\omega_i}(x_i), x_{i+1}) < \delta'(f_{\omega_i}(x_i))$ implies that

$$d(g_{\omega_i}(h^{-1}(x_i)), h^{-1}(x_{i+1})) = d(h^{-1}(f_{\omega_i}(x_i)), h^{-1}(x_{i+1})) < \delta(g_{\omega_i}(h^{-1}(x_i))).$$

It yields that $Np^c(\{x_i\}_{i\geq 0}, F, \omega, \delta') \subset Np^c(\{h^{-1}(x_i)\}_{i\geq 0}, G, \omega, \delta)$ and so $\{h^{-1}(x_i)\}_{i\geq 0}$ is a δ-ergodic pseudo orbit of G. Since the G has the ergodic shadowing property, there exist $z \in X$ and $\varphi \in \Sigma^m$ with $\varphi_i = \omega_i$ for $i \in Np^c(\{h^{-1}(x_i)\}_{i\geq 0}, G, \omega, \delta)$, such that $Ns(\{h^{-1}(x_i)\}_{i\geq 0}, G, \varphi, z, \epsilon)$ has zero density. Since for any $i \in Ns^c(\{h^{-1}(x_i)\}_{i\geq 0}, G, \varphi, z, \epsilon)$, $d(g^i_\varphi(z), h^{-1}(x_i)) < \epsilon(g^i_\varphi(z))$, we have

$$d'(h(g^i_\varphi(z)), x_i) = d'(f^i_\varphi(h(z)), x_i) < \epsilon'(f^i_\varphi(h(z))).$$

This means that $Ns^c(\{h^{-1}(x_i)\}_{i\geq 0}, G, \varphi, z, \epsilon) \subset Ns^c(\{x_i\}_{i\geq 0}, F, \varphi, h(z), \epsilon')$, which implies that $h(z)$, ϵ'-ergodic shadows $\{x_i\}_{i\geq 0}$. Thus F the has the ergodic shadowing property. □

The proof of the next lemma for a semigroup G on a compact metric space X was appeared in [4]. In the following theorem, shows that it holds for semigroup G on non-compact metric spaces with the new notions for δ-chain and shadowing property introduced in Definitions 2.1 and 2.2.

THEOREM 2.6. *Let (X,d) be a metric space and G be a semigroup associated with finite family $\{id, g_1,\dots,g_m\}$ of continuous maps on X. If the semigroup G has the shadowing property, then G is topologically mixing if and only if it is chain mixing.*

References

1. A.Z. Bahabadi, *Shadowing and average shadowing properties for iterated function systems*, Georgian Math. J. 22 (2015), 179–184.
2. A. Fakhari and F. H. Ghane, *On shadowing: ordinary and ergodic*, J. Math. Anal. Appl. 364 (2010), 151–155.
3. K. Lee, N. Nguyen, and Y. Yang, *Topological stability and spectral decomposition for homeomorphisms on noncompact spaces*, Discrete Contin. Dyn. Syst. 38 (2018), 2487–2503.
4. Z. Shabani, *Ergodic Shadowing of Semigroup Actions*, Bull. Iran. Math. Soc. 46 (2020), 303–321.
5. Z. Shabani, A. Barzanouni and X. Wu, *Recurrent sets and shadowing for finitely generated semigroup actions on metric spaces*, Hacet. J. Math. Stat. 50 (2021) 934–948.