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Conjugate of dynamical systems in locales

Ali Akbar Estaji[1,*], Maryam Robat Sarpoushi[2]

[1,2]Faculty of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, Iran

Abstract. For a dynamical system (X, f), the concept of "conjugate" is studied by many authors. Our goal is to introduce and study this concept in pointfree topology to give a description of categorical properties of conjugate in localic dynalical systems. We give the relation between the category **TDS** of dynamical systems and the category **LDS** of localic dynamical sysytems and continuous maps between them.

Keywords: dynamical system, locale, functor, adjunction.

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1. Introduction

In this section we briefly touch category theory and give the definitions and results needed in the next section. For more details, the reader can see [5] and [4].

Definition 1.1. Let $F, G : \mathbb{C} \to \mathbb{D}$ be functors. A *natural transformation* $\tau : F \to G$ is a function $\tau : Obj\mathbb{C} \to \mathrm{Mor}\mathbb{D}$ assigning to each object A in $\mathbb{C}$ a morphism $\tau_A : FA \to GA$ in $\mathbb{D}$ such that for every morphism $f : A \to B$ in $\mathbb{C}$, $(Gf)(\tau_A) = \tau_A(Ff)$. In this case we denote τ by $(\tau_A)_{A\in\mathbb{C}}$, and call each τ_A a component of τ.

Let $G : \mathbb{D} \to \mathbb{C}$ be a functor and $A \in Obj\mathbb{C}$. A *universal arrow* from A to G is an object B of D together with a morphism $u : A \to GB$ in $\mathbb{D}$ such that for each $D \in Obj\mathbb{D}$ and each morphism $f : A \to GD$ in $\mathbb{C}$ there is a unique morphism $\bar{f} : B \to D$ in $\mathbb{D}$ with $(G\bar{f}) \circ u = f$. Dualizing this, we get the notion of a couniversal arrow from A to G.

Definition 1.2. Let $G : \mathbb{D} \to \mathbb{C}$ be a functor. A *left adjoint* to G is a functor $F : \mathbb{C} \to \mathbb{D}$ such that for each $A \in Obj\mathbb{C}$, $B \in Obj\mathbb{D}$ there is an isomorphism $\alpha_{A,B} : Hom_{\mathbb{D}}(FA, B) \to Hom_{\mathbb{C}}(A, GB)$ which is natural in A and B (that is, $(\alpha_{A,-})_{A\in Obj\mathbb{C}} : Hom(FA, -) \to Hom(A, G-)$ and $(\alpha_{-,B})_{B\in Obj\mathbb{D}} : Hom(F-, B) \to Hom(-, GB)$ are natural isomorphisms). In this case we write $F \dashv G$, and say (F, G, α) is an *adjunction*, or also G is a *right adjoint* to F.

Theorem 1.3. *For functors $F : \mathbb{C} \to \mathbb{D}$ and $G : \mathbb{D} \to \mathbb{C}$, the following are equivalent:*

*Speaker. Email address: aaestaji@hsu.ac.ir; aaestaji@gmail.com,

(a) *F is a left adjoint to G.*
(b) *There exists a natural transformation $\eta : I_{\mathbb{C}} \to G \circ F$, called unit or front adjunction, such that for each $A \in Obj\mathbb{C}, \eta_A : A \to GFA$ is a universal arrow from A to G.*
(c) *There exists a natural transformation $\epsilon : F \circ G \to I_{\mathbb{D}}$, called counit or back adjunction, such that for each $B \in Obj\mathbb{D}$, $\epsilon_B : FGB \to B$ is a couniversal arrow from B to F.*

Our references for frames and locales are [4] and [5].

A *frame* (or *locale*) is a complete lattice L in which the infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \colon s \in S\}$$

holds for all $a \in L$ and $S \subseteq L$. We denote by 0 and 1, respectively, the bottom and top elements of L. A *frame homomorphism* (or *frame map*) is a map between frames which preserves finite meets, including the top element, and arbitrary joins, including the bottom element. An element $p \in L$ is said to be prime if $p < 1$ and $x \wedge y \leq p$ implies $a \leq p$ or $b \leq p$. Recall the contravariant functor Σ from **Frm** to the category **Top** of topological spaces which assigns to each frame L its *spectrum* ΣL of prime elements with $\Sigma_a = \{p \in \Sigma L \colon a \not\leq p\}$ $(a \in L)$ as its open sets. Also, we have $\Sigma L = \{f \colon L \to \mathbf{2} \colon f$ is a frame map $\}$, where $\mathbf{2} = \{0, 1\}$. The frame of all open sets of a topological space X is denoted by $\mathfrak{O}(X)$. The right adjoint of a frame homomorphism $h \colon L \to M$ is denoted by h_*. We shall use the terms "frame" and "locale" interchangeably, but when we wish to consider frame homomorphisms we will rather use "frame".

A *localic map* $f \colon L \to M$ is a meet-preserving mapping between locales the left adjoint of which preserves binary meets.

For the category **Loc**, we have the functor Lc: **Top** $\to$ **Loc** by setting $\mathrm{Lc}(X) = \mathfrak{O}(X)$ and $\mathrm{Lc}(f) = \mathfrak{O}(f)_*$ where $\mathfrak{O}(f) = f^{-1}$.

A discrete-time dynamical system (X, f) is a continuous map f on a nonempty topological space X, i.e. $f : X \to X$. The dynamics is obtained by iterating the map. The reader can see [1, 2] and [3] for more informations on dynamical systems.

Definition 1.4. A homomorhism of dynamical sysytems from (X, f) to (Y, g) is a map $\varphi : X \to Y$ such that $\varphi \circ f = g \circ \varphi$. In this case, we say that two dynamical systems are conjugated.

Adapting by this, we call (L, f) a locallic dynamical system if L is a locale and $f : L \to L$ is a locallic map.

2. Main results

Recall that for dynamical systems (X, f) and (Y, g), we call continuous function $\varphi : X \to Y$ a *morphism* if the following diagram commuts.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & X \arrow[d, "\varphi"] \\ Y \arrow[r, "f"'] & Y \end{tikzcd}$$

We denote the category of all topological dynamical systems and their morphisms by **TDS**. Now, we give the next definition.

Definition 2.1. A *localic dynamical system* (L, f) is a localic map f on a locale L, that is, $f\colon L \to L$.

Definition 2.2. Let (L, f) and (M, g) be localic dynamical systems. We call continuous function $h : L \to M$ a *morphism* if the following diagram commutes.

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "h"'] & L \arrow[d, "h"] \\ M \arrow[r, "f"'] & M \end{tikzcd}$$

We denote the category of all localic dynamical systems and their morphisms by **LDS**. Here, we are going to give a relation between **TDS** and **LDS**.

Foe this, we define $\Sigma : \mathbf{LDS} \longrightarrow \mathbf{TDS}$ with $(L, f) \longmapsto (\Sigma L, \Sigma f^*)$. It is clear that if $h : L \to M$ be a localic map such that $h \in Hom\big((L, f), (M, g)\big)$, then $g \circ h = h \circ f$. This implies that $\Sigma h^* \circ \Sigma g^* = \Sigma f^* \circ \Sigma h^*$ which means that

$$\begin{tikzcd} \Sigma M \arrow[r, "\Sigma g^*"] \arrow[d, "\Sigma h^*"'] & \Sigma M \arrow[d, "\Sigma h^*"] \\ \Sigma L \arrow[r, "\Sigma f^*"'] & \Sigma L \end{tikzcd}$$

and therefore $\Sigma h^* \in Hom\big((\Sigma M, \Sigma g^*), (\Sigma L, \Sigma f^*)\big)$. Threfore we have:

Proposition 2.3. *The functor Σ is a countravariant functor from **LDS** to **TDS**.*

Now, we define $\mathrm{Lc} : \mathbf{TDS} \longrightarrow \mathbf{LDS}$ with $(X, f) \longmapsto (\mathrm{Lc}(X), \mathrm{Lc}(f))$. It is clear that if $h : X \to Y$ be a morphism such that $h \in Hom\big((X, f), (Y, g)\big)$, then $g \circ h = h \circ f$. This implies $\mathrm{Lc}(h) \circ \mathrm{Lc}(f) = \mathrm{Lc}(g) \circ \mathrm{Lc}(h)$, which means that the next diagram commuts

$$\begin{tikzcd} \mathrm{Lc}(X) \arrow[r, "\mathrm{Lc}(f)"] \arrow[d, "\mathrm{Lc}(h)"'] & \mathrm{Lc}(X) \arrow[d, "\mathrm{Lc}(h)"] \\ \mathrm{Lc}(Y) \arrow[r, "\mathrm{Lc}(g)"'] & \mathrm{Lc}(Y) \end{tikzcd}$$

and therefore $\mathrm{Lc}(h) \in Hom\big((\mathrm{Lc}(X), \mathrm{Lc}(f)), (\mathrm{Lc}(Y), \mathrm{Lc}(g))\big)$. Thus we have:

Proposition 2.4. *The functor Lc is a covariant functor from **TDS** to **TDS**.*

Now, Put $\lambda : I_{\mathbf{TDS}} \to \Sigma\mathrm{Lc}$ such that, for every X, $\lambda_X : X \longrightarrow \Sigma Lc(X)$ is gicen by $x \longmapsto \{U \in \mathfrak{O}(X) : x \in U\}$. Now, let $h : (X, f) \to (Y, g)$ be a morphism in **TDS**, then $g \circ h = h \circ f$. This implies $\Sigma\mathrm{Lc}(h) \circ \lambda_X = \lambda_Y \circ h$ and so the following diagram is commuted:

$$\begin{tikzcd} X \arrow[r, "\lambda_X"] \arrow[d, "h"'] & \Sigma\mathrm{Lc}(X) \arrow[d, "\Sigma\mathrm{Lc}(h)"] \\ Y \arrow[r, "\lambda_Y"'] & \Sigma\mathrm{Lc}(Y) \end{tikzcd}$$

We put $\varphi : L \longrightarrow \mathrm{Lc}\Sigma L$ with $a \longmapsto \{F : a \in F\ ,\ F \text{ is a completely prime filter on } L\}$. It is clear that φ is a frame map. Now, we define $\sigma : \mathrm{Lc}\Sigma \to I_{\mathbf{LDS}}$ and set $\sigma_L = (\varphi_L)_*$. Now, let $f : X \to X$ be a continuous function, then, for a completely prime filter F, we have $(\Sigma\mathrm{Lc}(f))(F) = (\mathrm{Lc}(f)^*)^{-1}(F)$ This implies that, for $h : X \to Y$, we have $\Sigma\mathrm{Lc}(h)\lambda = \lambda_Y h$.

Proposition 2.5. *The function λ_X is a natural transformation.*

Now, let $f : L \to M$ be a localic map. We have

$$\begin{tikzcd}
(\mathrm{Lc}\Sigma)(L) \arrow[r, "\sigma_L"] \arrow[d, "(\mathrm{Lc}\Sigma)(f)"'] & L \arrow[d, "f"] \\
(\mathrm{Lc}\Sigma)(M) \arrow[r, "\sigma_M"'] & M
\end{tikzcd}$$

and so $f \circ \sigma_L = \sigma_M \circ (\mathrm{Lc}\Sigma)(f)$ which implies that $\sigma_L^* \circ f^* = (\mathrm{Lc}\Sigma)(f)^* \circ \sigma_M^*$.

Let $\lambda : I_{\mathbf{TDS}} \to \Sigma\mathrm{Lc}$. For every $(X, f) \in Obj(\mathbf{TDS})$, we have $\lambda_{(X,f)} : (X, f) \to \Sigma\mathrm{Lc}(X, f)$ where $\lambda_{(X,f)} = \lambda_X : X \to \Sigma\mathrm{Lc}(X)$ is a continuous fuction. It is easy to see that, for every $x \in X$, we have $\Sigma(\mathrm{Lc}(f))\lambda_{(X,f)}(x) = \lambda_{(X,f)}f(x)$ which shows that the following diagram is commuted:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\lambda_{(X,f)}"'] & X \arrow[d, "\lambda_{(X,f)}"] \\
\Sigma\mathrm{Lc}(X) \arrow[r, "\Sigma(\mathrm{Lc}(f))"'] & \Sigma\mathrm{Lc}(X)
\end{tikzcd}$$

Now, let $h : (X, f) \to (Y, g)$ be a morphism in **TDS**. Thus, for every $x \in X$, we have $\Sigma(\mathrm{Lc}(h)) \circ \lambda_{(X,f)}(x) = \lambda_{(Y,g)} \circ h(x)$ which shows that the following diagram commuts.

$$\begin{tikzcd}
(X, f) \arrow[r, "\lambda_{(X,f)}"] \arrow[d, "h"'] & (\Sigma(\mathrm{Lc}(X)), \Sigma(\mathrm{Lc}(f))) \arrow[d, "\lambda_{(X,f)}"] \\
(Y, g) \arrow[r, "\lambda_{(Y,g)}"'] & (\Sigma(\mathrm{Lc}(Y)), \Sigma(\mathrm{Lc}(g)))
\end{tikzcd}$$

For every (L, f), we define $\sigma_{(L,f)} : (\mathrm{Lc}\Sigma(L), \mathrm{Lc}\Sigma(f)) \to (L, f)$ where $\sigma_{(L,f)} = (\varphi)_* : \mathrm{Lc}\Sigma L \to L$ is a localic map. One can see that $\sigma_{(L,f)} \circ \mathrm{Lc}\Sigma f = f \circ \sigma_{(L,f)}$ which shows that the following diagram commuts.

$$\begin{tikzcd}
\mathrm{Lc}\Sigma L \arrow[r, "\mathrm{Lc}\Sigma f"] \arrow[d, "\sigma_{(L,f)}"'] & \mathrm{Lc}\Sigma L \arrow[d, "\sigma_{(L,f)}"] \\
L \arrow[r, "f"'] & L
\end{tikzcd}$$

Now, let $h : (L, f) \to (M, g)$ be a morphism in **LDS**. We have $\sigma_{(M,g)} \circ \mathrm{Lc}\Sigma h = h \circ \sigma_{(L,f)}$ which shows that the following diagram commuts.

$$\begin{tikzcd}
(\mathrm{Lc}\Sigma L, \mathrm{Lc}\Sigma f) \arrow[r, "\sigma_{(L,f)}"] \arrow[d, "\mathrm{Lc}\Sigma L"'] & (L, f) \arrow[d, "h"] \\
(\mathrm{Lc}\Sigma M, \mathrm{Lc}\Sigma g) \arrow[r, "\sigma_{(M,g)}"'] & (M, g)
\end{tikzcd}$$

Proposition 2.6. *The function σ is a natural transformation.*

3. Conclusion

Corollary 3.1. *Let Σ : **LDS** $\to$ **TDS** and Lc : **TDS** $\to$ **LDS** be as the same in the previous section. Then* $\mathrm{Lc} \dashv \Sigma$.

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