
# Mean curvature of semi-symmetric metric connections

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Abstract. In this paper, we first introduce semi-symmetric metric and semi-symmetric non-metric connections on a $(n+p)$-dimensional semi-Riemannian manifold $(M, g)$. Then we obtain relations between mean curvatures of these connections and the Levi-Civita connection.

**Keywords:** Mean curvature, Semi-symmetric metric connection, semi-symmetric non metric connection.

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## 1. Introduction

The notion of semi-symmetric connections on a differentiable manifold is introduced by Friedman and Schouten in 1924 [3]. The study of it was further developed by some researcher such as Yano [6] and Chaki and Konar [2].
These mentioned connections have applications in Physics. There are various physical problems involving them.
In this work, we find relations between mean curvatures of these connections and the Levi-Civita connection.
In the following, we provide basic information used in the paper.
A linear connection $\nabla$ on a semi-Riemannian manifold $(M, g)$ is said to be semi-symmetric if the torsion tensor $T$ of the connection $\nabla$ satisfies

$$T(X, Y) = \omega(Y)X - \omega(X)Y, \tag{1}$$

for any vector fields $X, Y$ on $M$ and $\omega$ is a 1-form given by $\omega(X) = g(X, W)$, where $W$ is the vector field associated with the 1-form $\omega$.
If $\nabla g = 0$, then the connection $\nabla$ is said to be a metric connection; otherwise, it is called non-metric [4].
Now let $(M, g)$ be an $(n + p)-$dimensional semi-Riemannian manifold endowed with an

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$n-$distribution $\mathcal{D}$. The real vector space of all symmetric bilinear mappings $g_x : \mathcal{D}\times\mathcal{D} \longrightarrow \mathbb{R}$, is denoted by $L_s^2(\mathcal{D}_x, \mathbb{R})$. Then we consider the vector bundle

$$L_s^2(\mathcal{D}, \mathbb{R}) = \bigcup_{x\in M} L_s^2(\mathcal{D}_x, \mathbb{R})$$

over $M$. The metric tensor $g$ induces a global section of $L_s^2(\mathcal{D}, \mathbb{R})$ which is denoted by the same symbol $g$. If $g$ is non-degenerate, the pair $(\mathcal{D}, g)$ is a semi-Riemannian distribution. Also the vector bundle $D^{\perp}$ is considered as follows

$$\mathcal{D}^{\perp} = \bigcup_{x\in M} \mathcal{D}_x^{\perp},$$

where $\mathcal{D}_x^{\perp}$ is the complementary orthogonal subspace to $\mathcal{D}_x$ in $(T_xM, g_x)$. The metric tensor $g$ induces a semi-Riemannian metric on $\mathcal{D}^{\perp}$, and here again we denote it by $g$. Therefore $(\mathcal{D}^{\perp}, g)$ is a semi-Riemannian distribution. Thus we have

$$TM = \mathcal{D} \oplus \mathcal{D}^{\perp}. \tag{2}$$

Also the mappings $\mathcal{P}$ and $\mathcal{Q}$ are the projection morphisms of $TM$ on $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively.

$$a)\ \tilde{H}(X, \mathcal{P}Y) = \mathcal{Q}\tilde{\nabla}_X\mathcal{P}Y, \quad b)\ \tilde{H}^{\perp}(X, \mathcal{Q}Y) = \mathcal{P}\tilde{\nabla}_X\mathcal{Q}Y, \tag{3}$$

where $\tilde{H}$ and $\tilde{H}^{\perp}$ are the second fundamental forms of $\mathcal{D}$ and $\mathcal{D}^{\perp}$ with respect to $\tilde{\nabla}$, respectively [1].

## 2. Main Results

We now suppose that the semi-Riemannian manifold $(M, g)$ admits a semi-symmetric metric connection given by

$$\nabla_X Y = \tilde{\nabla}_X Y + \omega(Y)X - g(X, Y)W, \tag{4}$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $(M, g)$, $\omega$ is a 1-form and $W$ is the vector field defined by

$$g(W, X) = \omega(X), \tag{5}$$

for any vector field $X$ of $M$ (see [5], [6]).

$$a)\ H(X, \mathcal{P}Y) = \mathcal{Q}\nabla_X\mathcal{P}Y, \quad b)\ H^{\perp}(X, \mathcal{Q}Y) = \mathcal{P}\nabla_X\mathcal{Q}Y. \tag{6}$$

We call $H$ (resp. $H^{\perp}$) the second fundamental forms of $\mathcal{D}$ (resp. $\mathcal{D}^{\perp}$) with respect to $\nabla$. Since $\nabla$ is metric by using (**??**a) and (**??**b) we obtain that

$$\begin{aligned} g(H(X, \mathcal{P}Y), \mathcal{Q}Z &= g(\nabla_X\mathcal{P}Y, \mathcal{Q}Z) \\ &= -g(\mathcal{P}Y, \nabla_X\mathcal{Q}Z) \\ &= -g(H^{\perp}(X, \mathcal{Q}Z), \mathcal{P}Y). \end{aligned} \tag{7}$$

By defination of the semi-symmetric metric connection $\nabla$ and by (3a), (3b), (6a) and (6b) we deduce that

$$H(X, \mathcal{P}Y) = \tilde{H}(X, \mathcal{P}Y) + \omega(\mathcal{P}Y)\mathcal{Q}X - g(X, \mathcal{P}Y)\mathcal{Q}W,$$

and

$$H^{\perp}(X, \mathcal{Q}Y) = \tilde{H}^{\perp}(X, \mathcal{Q}Y) + \omega(\mathcal{Q}Y)\mathcal{P}X - g(X, \mathcal{Q}Y)\mathcal{P}W.$$

Therefore

$$H(\mathcal{P}X,\mathcal{P}Y)=\tilde{H}(\mathcal{P}X,\mathcal{P}Y)-g(\mathcal{P}X,\mathcal{P}Y)\mathcal{Q}W, \tag{8}$$

$$H^{\perp}(\mathcal{Q}X,\mathcal{Q}Y)=\tilde{H}^{\perp}(\mathcal{Q}X,\mathcal{Q}Y)-g(\mathcal{Q}X,\mathcal{Q}Y)\mathcal{P}W. \tag{9}$$

The second fundamental forms and the shap operators of the distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ are related by [1]

$$g(\tilde{H}(\mathcal{P}X,\mathcal{P}Y),\mathcal{Q}Z)=g(\tilde{A}_{\mathcal{Q}Z}\mathcal{P}X,\mathcal{P}Y), \tag{10}$$

and

$$g(\tilde{H}^{\perp}(\mathcal{Q}X,\mathcal{Q}Y),\mathcal{P}Z)=g(\tilde{A}^{\perp}_{\mathcal{P}Z}\mathcal{Q}X,\mathcal{Q}Y). \tag{11}$$

Let $E_1,...,E_n$ be an orthonormal basis for $\mathcal{D}_p$ of signature $\varepsilon_1,...,\varepsilon_n$, then

$$\tilde{\text{II}}=\frac{1}{n}\Sigma_{i=1}^{n}\varepsilon_i\tilde{H}(E_i,E_i)$$

is the mean curvature vector field of $(\mathcal{D},g)$ with respect to $\tilde{\nabla}$ and

$$\text{II}=\frac{1}{n}\Sigma_{i=1}^{n}\varepsilon_i H(E_i,E_i)$$

is the mean curvature of $(\mathcal{D},g)$ with respect to $\nabla$. We obtain from (8)

$$\begin{aligned}
\text{II} &= \frac{1}{n}\Sigma_{i=1}^{n}\varepsilon_i H(E_i,E_i) && (12)\\
&= \frac{1}{n}\Sigma_{i=1}^{n}\varepsilon_i\{\tilde{H}(E_i,E_i)-g(E_i,E_i)\mathcal{Q}W\} && (13)\\
&= \frac{1}{n}\Sigma_{i=1}^{n}\varepsilon_i\{\tilde{H}(E_i,E_i)\}-\frac{1}{n}\Sigma_{i=1}^{n}\varepsilon_i g(E_i,E_i)\mathcal{Q}W\}\\
&= \tilde{\text{II}}-\mathcal{Q}W.
\end{aligned}$$

We can state the following. The mean curvature of $(\mathcal{D},g)$ with respect to $\tilde{\nabla}$ concides with that of $(\mathcal{D},g)$ with respect to $\nabla$, if a vector $W$ lies in $\mathcal{D}$.

If $\tilde{H}$ vanishes, then $(\mathcal{D},g)$ is totally geodesic with respect to $\tilde{\nabla}$, and if $\tilde{H}(\mathcal{P}X,\mathcal{P}Y)=g(\mathcal{P}X,\mathcal{P}Y)\tilde{\text{II}}$, then $(\mathcal{D},g)$ is totally umblical with respect to $\tilde{\nabla}$. Similarly, If $H$ vanishes, then $(\mathcal{D},g)$ is totally geodesic with respect to $\nabla$, and if $H(\mathcal{P}X,\mathcal{P}Y)=g(\mathcal{P}X,\mathcal{P}Y)\text{II}$, then $(\mathcal{D},g)$ is totally umblical with respect to $\nabla$.
From (8) and (12), we have the following Proposition: The semi-symmetric distribution $(\mathcal{D},g)$ is totally umblical with respect to $\tilde{\nabla}$ if and only if it is totally umblical with respect to the semi-symmetric metric connection $\nabla$. If $\tilde{\text{II}}=0$ ($resp.\text{II}=0$), then $(\mathcal{D},g)$ is called minimal with respect to $\tilde{\nabla}$ ($resp.\nabla$) from equation (12) we have the following result:

THEOREM 2.1. $(\mathcal{D},g)$ *is minimal with respect to the semi-symmetric metric connection* $\nabla$ *if and only if it is minimal with respect to the Levi-Civita connection, when a vector field* $W$ *lies in* $\mathcal{D}$.

A linear connection $\breve{\nabla}$ on a semi-Riemannian manifold $(M,g)$ defined by

$$\breve{\nabla}_X Y=\tilde{\nabla}_X Y+\omega(Y)X \tag{14}$$

is a semi-symmetric non-metric connection, where $\tilde{\nabla}$ is the Levi-Civita connection of $(M,g)$ and $\omega$ is a 1-form.

$$a)\ \breve{H}(X,\mathcal{P}Y)=\mathcal{Q}\breve{\nabla}_X\mathcal{P}Y,\quad b)\ \breve{H}^{\perp}(X,\mathcal{Q}Y)=\mathcal{P}\breve{\nabla}_X\mathcal{Q}Y. \tag{15}$$

$\breve{H}$ and $\breve{H}^{\perp}$ are the second fundamental forms of $\mathcal{D}$ and $\mathcal{D}^{\perp}$ respectively. By (14) and (15)

$$\breve{H}(X, \mathcal{P}Y) = \tilde{H}(X, \mathcal{P}Y) + \omega(\mathcal{P}Y)\mathcal{Q}X, \tag{16}$$

$$\breve{H}^{\perp}(X, \mathcal{Q}Y) = \tilde{H}^{\perp}(X, \mathcal{Q}Y) + \omega(\mathcal{Q}Y)\mathcal{P}X. \tag{17}$$

Thus we obtain that

$$a)\ \breve{H}(\mathcal{P}X, \mathcal{P}Y) = \tilde{H}(\mathcal{P}X, \mathcal{P}Y) \quad b)\ \breve{H}^{\perp}(\mathcal{Q}X, \mathcal{Q}Y) = \tilde{H}^{\perp}(\mathcal{Q}X, \mathcal{Q}Y). \tag{18}$$

$$\breve{\amalg} = \frac{1}{n}\Sigma_{i=1}^{n}\varepsilon_i \breve{H}(E_i, E_i) \tag{19}$$

is the mean curvature of $(\mathcal{D}, g)$ with respect to the semi-symmetric non-metric connection $\breve{\nabla}$, where $E_1, ..., E_n$ is an orthonormal basis for $\mathcal{D}_p$ of signature $\varepsilon_1, ..., \varepsilon_n$.
Hence by 16 we may state the following results.

THEOREM 2.2. *The mean curvature of* $(\mathcal{D}, g)$ *with respect to* $\tilde{\nabla}$ *concides with that of* $(\mathcal{D}, g)$ *with respect to the semi-symmetric non-metric connection* $\breve{\nabla}$. If $\breve{\amalg} = 0$, then $(\mathcal{D}, g)$ is called minimal with respect to $\breve{\nabla}$. $(\mathcal{D}, g)$ is minimal with respect to $\breve{\nabla}$ if and only if it is minimal with respect to $\tilde{\nabla}$.

If $\breve{H}$ vanishes, then $(\mathcal{D}, g)$ is totally geodesic with respect to $\breve{\nabla}$ and if $\breve{H}(\mathcal{P}X, \mathcal{P}Y) = g(\mathcal{P}X, \mathcal{P}Y)\breve{\amalg}$, then $(\mathcal{D}, g)$ is said to be totally umblical with respect to $\breve{\nabla}$. The next Theorem follows directly from Theorem 2.2.

THEOREM 2.3. *The semi-Riemannian distribution* $(\mathcal{D}, g)$ *is totally geodesic (totally umblical) with respect to* $\tilde{\nabla}$ *if and only if it is totally geodesic (totally umblical) with respect to* $\breve{\nabla}$.

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