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Analysis of a nonlinear election model in fractional order

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Abstract. This article is devoted to studying a fractional model of the election with three parties. For this aim, we used the fractional derivative in Caputo sense. The stability of some equlibria in this model is investigated by the fractional RouthHurwitz stability criterion. Also, our numerical results of the proposed model show the simplicity and the efficiency of the proposed criterion.

Keywords: Fractional-order nonlinear election model , Population dynamics, Stability.

AMS Mathematics Subject Classification [2020]: 91F10, 92D25, 34D20

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1. Introduction

The 2000 and 2016 United States presidential elections show us political parties shape public opinion, but their influence is limited. Indeed, we should be noted that although many countries have dual political parties which play a main role in their elections, there is a special parameter which is people's opinions. Some researches show sometimes people change the mind of these political parties or maybe build a new political party. Indeed, a new party arises which a growing number of voters move into it. The 2000 and 2016 U.S. presidential elections and 2005 Iran presidential election are some samples of this idea. According to the importance of elections in the democracy of countries, our attention is attracted to study the movement of voters between political parties and the population dynamics amongst each group. For this aim, a nonlinear mathematical model in fractional order with a constant population assumption is considered.

In most countries, it is important that political independence is retained by the people and exercised directly by citizens. The usual mechanism in these countries is a decision-making process by which citizens who have necessary conditions choose an individual to hold formal office. In this approach, the influence of political parties on people's views is treated as a disease that person is affected. Therefore, it can be assumed that members move from one political party to another one when they are exposed to the ideology of

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other parties. Some modeling studies have been conducted regarding the growth of political parties and voters [2–6]. In the modeling process, we assume that the total population is constant, N. This total population has been divided into four classes:

(1) V: Population of eligible voters
(2) A: Population of Political Party A
(3) B: Population of Political Party B
(4) C: Population of Political Party C.

Bauelos et al. [5] introduced the following model after simplification

$$\frac{dX}{dt} = X(a(-X-Y-Z+1)-\mu-\psi Y+\Omega Z), \tag{1}$$

$$\frac{dY}{dt} = Y(b(-X-Y-Z+1)-\mu+\psi(X-Z)), \tag{2}$$

$$\frac{dZ}{dt} = Z(c(-X-Y-Z+1)-\mu-X\Omega+\psi Y), \tag{3}$$

which

- X : Proportion of Political Party A
- Y : Proportion of Political Party B
- Z : Proportion of Political Party C
- a : per capita recruitment rate of Party A from V
- b : per capita recruitment rate of Party B from V
- c : per capita recruitment rate of Party$C$ from V
- μ : rate at which individuals enter and leave voting system
- Ω : Net Shift between Party A and Party C
- ψ : Net Shift between Party A and Party B and Net Shift between Party B and Party C.

In the next Section, we consider the fractional order form of this model as follows

$$D^{\alpha}X(t) = X(a(-X-Y-Z+1)-\mu-\psi Y+\Omega Z), \tag{4}$$

$$D^{\alpha}Y(t) = Y(b(-X-Y-Z+1)-\mu+\psi(X-Z)), \tag{5}$$

$$D^{\alpha}Z(t) = Z(c(-X-Y-Z+1)-\mu-X\Omega+\psi Y), \tag{6}$$

to investigate its dynamics by determining equilibrium points of this system analytically and discuss their stability.

2. Main results

The equilibrium points of our model are denoted by (X^*, Y^*, Z^*), where $X^* = \frac{A^*}{N}$, $Y^* = \frac{B^*}{N}$, $Z^* = \frac{C^*}{N}$, and $V^* = N - (A^* + B^* + C^*)$. These equilibria will be in the one of four forms

- the party-free equilibria
- the single-party equilibria
- the dual-party equilibria
- the interior equilibria

TABLE 1. Equilibrium Points

case	equilibrium point	value	status
1	p1	$X = 0, Y = 0, Z = 1$	—
2	p2	$X = 0, Y = 1, Z = 0$	—
3	p3	$X = 0, Y = -1, Z = 1$	not acceptable
4	p4	$X = \frac{0.99}{\Omega}, Y = 0, Z = \frac{-0.99}{\Omega}$	not acceptable
5	p5	$X = \frac{0.99}{(2+\Omega)}, Y = \frac{0.99\Omega}{(2+\Omega)}, Z = \frac{0.99}{(2+\Omega)}$	—
6	p6	$X = 0., Y = 0., Z = 0.$	—
7	p7	$X = 1, Y = 0., Z = 0.$	—
8	p8	$X = -1, Y = 1, Z = 0.$	not acceptable

which is obtained in [3] as it is summarized in Table 1 On the other hand, it is not hard to compute the characteristic equation of the equilibria as the following polynomial:

$$\phi(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 \tag{7}$$

Now, expressing the discriminant of $\phi(\lambda)$ as

$$D(\phi) = 18a_1a_2a_3 + (a_1a_2)^2 - 4a_3a_1^2 - 4a_2^2 - 27a_3^2 \tag{8}$$

and using the result of Ahmed et al. [1], following fractional RouthHurwitz conditions associated with are observed:

(1) If $D(\phi) > 0$, then the necessary and sufficient condition for the equilibrium point to be locally asymptotically stable is $a_1 > 0, a_3 > 0, a_1a_2 > a_3$;
(2) If $D(\phi) < 0, a_1 \geq 0, a_2 \geq 0, a_3 > 0$, then the equilibrium point is locally asymptotically stable for $\alpha < \frac{2}{3}$,
(3) If $D(\phi) < 0, a_1 < 0, a_2 < 0, \alpha > \frac{2}{3}$, then all roots of Eq. 7 satisfy the condition $|arg(\lambda_i)| < \alpha\frac{\pi}{2}, i = 1, 2, 3$

To obtain the equilibria of system (4)-(6), we consider the following parameter values which is used in [5]

$$\begin{aligned} a &= b = c = \psi = 1, \\ \mu &= 0.01. \end{aligned}$$

Therefore, if $\Omega = \psi$ we have the results in Table2

By applying fractional RouthHurwitz stability criterion, the results is summarized in Table 3 Now, we consider p_6. By Table 3 and 2, it is an unstable point, see Figure 1-3. The result for other points is similar.

3. Conclusion

This paper is devoted to implement the analytical and numerical method for studying a nonlinear election model in fractional-order. In this work, we interested to discuss and study the discriminant of characteristic equation of our model to investigate the stability of equilibria. Also, some numerical results is presented for one of equlibria, origin. Our analytical and numerical results showed that it was unstable.

TABLE 2. Characteristic Equation

case	equilibrium point	value	characteristic equation
1	p1	$X=0, Y=0, Z=1$	$\lambda^3+1.03\lambda^2-0.9797\lambda-1.0099$
2	p2	$X=0, Y=1, Z=0$	$\lambda^3+0.03\lambda^2-0.9997\lambda-0.009999$
3	p3	$X=0, Y=-1, Z=1$	not acceptable
4	p4	$X=\frac{0.99}{\Omega}, Y=0, Z=\frac{-0.99}{\Omega}$	not acceptable
5	p5	$X=\frac{0.99}{(2+\Omega)}, Y=\frac{0.99\Omega}{(2+\Omega)}, Z=\frac{0.99}{(2+\Omega)}$	$\lambda^3+0.99\lambda^2+0.3267\lambda+0.035937$
6	p6	$X=0., Y=0., Z=0.$	$\lambda^3-2.97\lambda^2+2.9403\lambda-0.970299$
7	p7	$X=1, Y=0., Z=0.$	$\lambda^3+0.99\lambda^2-0.9801\lambda-0.970299$
8	p8	$X=-1, Y=1, Z=0.$	not acceptable

TABLE 3. Fractional RouthHurwitz Stability Criterion

case	equilibrium point	$D(\phi)$
1	p1	-7.73824
2	p2	-3.97242
3	p3	not acceptable
4	p4	not acceptable
5	p5	-0.407442
6	p6	201.692
7	p7	-7.56213
8	p8	not acceptable

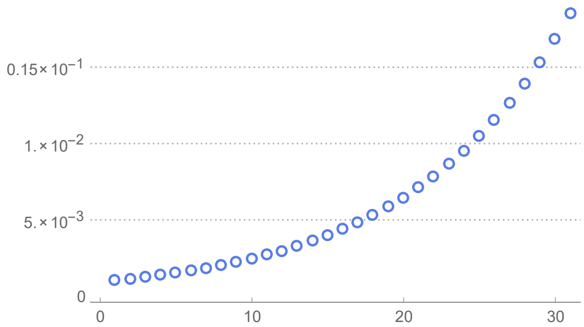

FIGURE 1. Behavior of the numerical solution X(t) using GABMM , $\alpha = 0.95$.

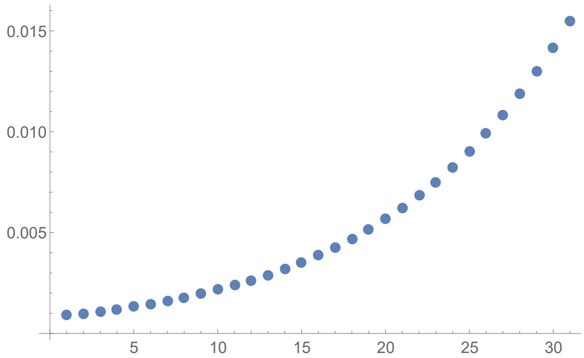

FIGURE 2. Behavior of the numerical solution Y(t) using GABMM , $\alpha = 0.95$.

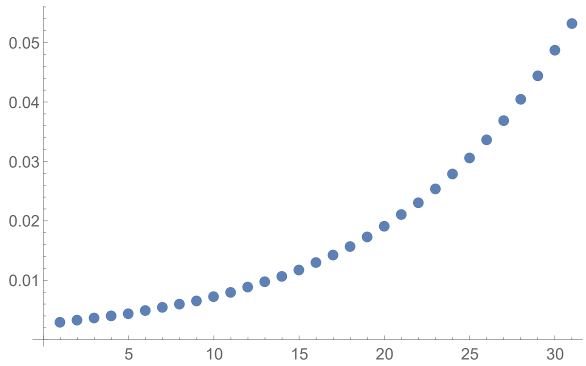

FIGURE 3. Behavior of the numerical solution Z(t) using GABMM , $\alpha = 0.95$.

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