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On Ricci curvature of Finsler warped product metrics

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Abstract. In this paper, we study a rich and important class of Finsler metrics called Finsler warped product metrics. We find an equation that characterizes locally projectively flat warped product metrics. Further, we study Einstein Finsler warped product metrics.

Keywords: Finsler warped product metrics, locally projectively flat, Einstein metrics

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1. Introduction

Finsler geometry is just Riemannian geometry without the quadratic restriction on its metrics [1]. For a Finsler metric $F = F(x, y)$, its geodesics curves are given by the system of differential equations $\ddot{c}^i + 2G^i(c, \dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients. A Finsler metric is called a Berwald metric if G^i are quadratic in $y \in T_xM$ for any $x \in M$.

The special Finsler metrics we are going to investigate are called Finsler warped product metrics which first introduced by Chen-Shen-Zhao and Kozma-Peter-Varga [2, 4]. By defnition, a Finsler warped product metric F on the product manifold $M := I \times \breve{M}$ where I is an interval of $\mathbb{R}$ and $\breve{M}$ is an $(n-1)$-dimensional manifold equipped with a Riemannian metric $\breve{\alpha}$ can be expressed in the following form:

$$F(u, v) := \breve{\alpha}(\breve{u}, \breve{v})\phi\Big(u^1, \frac{v^1}{\breve{\alpha}(\breve{u}, \breve{v})}\Big), \tag{1}$$

where $u = (u^1, \breve{u}), v = v^1\frac{\partial}{\partial u^1} + \breve{v}$ and ϕ is a suitable function defined on a domain of $\mathbb{R}^2$. This class of Finsler metrics concludes spherically symmetric Finsler metrics [2]. In [3], Gabrani-Rezaei-Sevim characterized the Finsler warped product metrics of isotropic

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Berwald curvature. Moreover, they studied the unicorn problem for the class of Finsler metrics. Troughout this paper, our index conventions are as follows:

$$1 \leq A \leq B \leq \ldots \leq n, \qquad 2 \leq i \leq j \leq \ldots \leq n.$$

We prove the following theorem:

THEOREM 1.1. *Let $F = \breve{\alpha}\phi(r,s), r = u^1, s = \frac{v^1}{\breve{\alpha}}$ be a warped product metric. Then F is locally projectively flat if and only if $\breve{\alpha}$ has constant sectional curvature κ ($\breve{\alpha}$ is locally projectively flat) and ϕ satisfies*

$$(\phi - s\phi_s)_r = 2[-\eta + \frac{2\eta' - \kappa}{4\eta}s^2]\phi_{ss}, \tag{2}$$

where $\eta = \eta(r)$ is a differential function.

Let $\eta(r)$ be a function such that the integrals

$$\int \frac{\kappa - 2\eta'}{2\eta}dr, \quad \int 2\eta e^{-\int \frac{\kappa-2\eta'}{2\eta}dr}dr$$

are well defined for $r \in R$. Then the general solution of (2) for $s > 0$ is [5]

$$\phi(r,s) = sh - s\int_{s_0}^{s} \sigma^{-2}\zeta\big[e^{-\int \frac{\kappa-2\eta'}{2\eta}dr}\sigma^2 + \int 2\eta e^{-\int \frac{\kappa-2\eta'}{2\eta}dr}dr\big]d\sigma, \tag{3}$$

where $s \in (0, s]$ and $h = h(r)$ and $\zeta = \zeta(r,\sigma)$ are arbitrary differentiable real functions.

The following result gives a characterization of Douglas Finsler warped product metrics to be Einstein in the case of two-dimensional Riemannian manifold $(\breve{M}, \breve{\alpha})$.

THEOREM 1.2. *Let $F = \breve{\alpha}\phi(r,s), r = u^1, s = \frac{v^1}{\breve{\alpha}}$ be a Douglas warped product metric on an n-dimensional manifold $M := I \times \breve{M}$. Then F has isotropic Ricci curvature*

$$Ric = (n-1)K(u)F^2$$

if and only if $\breve{\alpha}$ has constant Ricci curvature $(n-2)c, K(u) = K(r)$ and

$$(n-1)\Big\{\Psi^2 - [s\Psi_r - 2(\xi s^2 + \eta)\Psi_s] + c\Big\} + 2(2\eta\xi + \eta') - c = (n-1)K\phi^2, \tag{4}$$

where $\Psi = \frac{s\phi_r}{2\phi} - \frac{\phi_s}{\phi}[\xi(r)s^2 + \eta(r)]$ and $\xi = \xi(r)$ and $\eta = \eta(r)$ are two differential functions.

The following result gives a characterization of locally projectively flat Finsler warped product metrics to be Einstein in the case of two-dimensional Riemannian manifold $(\breve{M}, \breve{\alpha})$.

THEOREM 1.3. *Let $F = \breve{\alpha}\phi(r,s), r = u^1, s = \frac{v^1}{\breve{\alpha}}$ be a locally projectively flat warped product metric on an n-dimensional manifold $M := I \times \breve{M}$, where $\breve{\alpha}$ is Ricci flat, $c = 0$. Then F has isotropic Ricci curvature*

$$Ric = (n-1)K(u)F^2$$

if and only if the function ϕ satisfies the following PDE:

$$\Psi^2 - s\Psi_r + \frac{2\eta^2 - \eta' s^2}{\eta}\Psi_s = K\phi^2, \tag{5}$$

where $\Psi = \frac{s\phi_r}{2\phi} - \frac{\phi_s}{\phi}[\frac{2\eta^2 - \eta' s^2}{2\eta}]$ and $\eta = \eta(r)$ is a differential functions.

2. Preliminaries

Let F be a Finsler metric on an n-dimensional manifold M and G^A be the geodesic coefficients of F, which are defined by

$$G^A := \frac{1}{4}g^{AB}\{[F^2]_{u^C v^B}v^C - [F^2]_{u^B}\},$$

where $g_{AB}(u,v) = \left[\frac{1}{2}F^2\right]_{v^A v^B}$ and $(g^{AB}) = (g_{AB})^{-1}$.

Lemma 2.1. *The spray coefficients G^A of a Finsler warped product metric $F = \breve{\alpha}\phi(r,s)$ are given by [2]*

$$G^1 = \Phi\breve{\alpha}^2, \qquad G^i = \breve{G}^i + \Psi\breve{\alpha}^2\breve{l}^i, \tag{6}$$

where $\breve{l}^i = \frac{v^i}{\breve{\alpha}}$ and

$$\begin{cases} \Phi = \frac{s^2(\omega_r\omega_{ss} - \omega_s\omega_{rs}) - 2\omega(\omega_r - s\omega_{rs})}{2(2\omega\omega_{ss} - \omega_s^2)}, \\ \\ \Psi = \frac{s(\omega_r\omega_{ss} - \omega_s\omega_{rs}) + \omega_s\omega_r}{2(2\omega\omega_{ss} - \omega_s^2)}, \end{cases} \tag{7}$$

where $\omega = \phi^2$. Φ and Ψ can be rewritten as follows:

$$\Phi = s\,\Psi + A, \tag{8}$$

$$\Psi = \frac{s\phi_r}{2\phi} - \frac{\phi_s}{\phi}A, \tag{9}$$

where

$$A := \frac{s\phi_{rs} - \phi_r}{2\,\phi_{ss}}. \tag{10}$$

Moreover,

$$\mathbf{D} = D^A_{BCD}dx^B \otimes dx^C \otimes dx^D$$

is a tensor on $TM \setminus \{0\}$ which is called the Douglas tensor, where

$$D^A_{BCD} := \frac{\partial^3}{\partial v^B \partial v^C \partial v^D}\Big(G^A - \frac{1}{n+1}\frac{\partial G^C}{\partial v^C}v^A\Big). \tag{11}$$

A Finsler metric F is called Douglas metric if $\mathbf{D} = 0$. For a Berwald metrics, the spray coefficients G^i are quadratic in y. It follows that $\mathbf{D} = 0$, (11). The Berwald metrics are Douglas metric. H. Liu and X. Mo have proved that a warped product Finsler metric $F = \breve{\alpha}\phi(r,s)$ is of Douglas type if and only if

$$\Phi - s\Psi = \xi(r)s^2 + \eta(r),$$

where $\xi = \xi(r)$ and $\eta = \eta(r)$ are two differential functions, [5].

In order to prove the main theorems, we need the following lemmas:

Lemma 2.2. *A Finsler metric F on a manifold M (dim $M > 2$) is locally projectively flat if and only if $D = 0$ and $W = 0$.*

Lemma 2.3. *[6] Let $F = \breve{\alpha}\phi(r,s), r = u^1, s = \frac{v^1}{\breve{\alpha}}$ be a warped product metric. Then F is of scalar flag curvature if and only if $\breve{\alpha}$ has constant sectional curvature κ and*

$$\lambda - \nu = \kappa, \tag{12}$$

where

$$\lambda = (2\Phi_r - s\Phi_{rs}) + (2\Phi\Phi_{ss} - \Phi_s^2) + 2(\Phi_s - s\Phi_{ss})\Psi - (2\Phi - s\Phi_s)\Phi_s, \tag{13}$$

$$\mu = \Psi^2 - 2s\Psi\Psi_s - s\Psi_r + 2\Phi\Psi_s, \tag{14}$$
$$\tau = 2\Psi_r - s\Psi_{rs} + s(\Psi_s^2 - 2\Psi\Psi_{ss}) + 2\Psi_{ss}\Phi - \Psi_s\Phi_s, \tag{15}$$
$$\nu = s\tau + \mu. \tag{16}$$

In [2], Chen-Shen-Zhao obtained a formula for the Ricci curvature Ric of a Finsler warped product metric, and it is given at below.

Lemma 2.4. *[2] For a Finsler warped product metric $F = \breve{\alpha}\phi(r, s)$, the Ricci curvature Ric is given by*

$$Ric = \breve{Ric} + \breve{\alpha}^2[\lambda + (n-1)\mu - \nu], \tag{17}$$

where

$$\lambda = (2\Phi_r - s\Phi_{rs}) + (2\Phi\Phi_{ss} - \Phi_s^2) + 2(\Phi_s - s\Phi_{ss})\Psi - (2\Phi - s\Phi_s)\Phi_s, \tag{18}$$
$$\mu = \Psi^2 - 2s\Psi\Psi_s - s\Psi_r + 2\Phi\Psi_s, \tag{19}$$
$$\tau = 2\Psi_r - s\Psi_{rs} + s(\Psi_s^2 - 2\Psi\Psi_{ss}) + 2\Psi_{ss}\Phi - \Psi_s\Phi_s, \tag{20}$$
$$\nu = s\tau + \mu. \tag{21}$$

Lemma 2.5. *[6] Let $F = \breve{\alpha}\phi(r, s), r = u^1, s = \frac{v^1}{\breve{\alpha}}$ be a warped product metric on an n-dimensional manifold $M := I \times \breve{M}$. Then F has isotropic Ricci curvature*

$$Ric = (n-1)K(u)F^2$$

if and only if $\breve{\alpha}$ has constant Ricci curvature $(n-2)c, K(u) = K(r)$ and

$$(n-1)[K(r)\phi^2 - \mu] + \nu - \lambda = (n-2)c. \tag{22}$$

3. Proof of Main Theorems

Proof of the Theorem 1.1

We will prove this theorem by using Lemma 2.2. Let $F = \breve{\alpha}\phi(r, s), r = u^1, s = \frac{v^1}{\breve{\alpha}}$ be a warped product metric on an n-dimensional manifold $M := I \times \breve{M}$. F is of Douglas type if and only if

$$\Phi - s\Psi = \xi(r)s^2 + \eta(r), \tag{23}$$

where $\xi = \xi(r)$ and $\eta = \eta(r)$ are two differential functions, [5]. By (8), it is easy to see that (23) is equivalent to

$$A = \xi(r)s^2 + \eta(r). \tag{24}$$

According to Matsumoto's result, F is of vanishing Weyl curvature if and only if it is of scalar flag curvature. By (8), one can see that (12) is equivalent to

$$2\,AA_{ss} - sA_{rs} - A_s^2 + 2\,A_r - \kappa = 0. \tag{25}$$

By substituting (24) into (25), we get

$$4\,\xi\eta + 2\eta' - \kappa = 0. \tag{26}$$

By (10), (24), and (26), we obtain (2). This completes the proof. □

Proof of the Theorem 1.2 By Lemma 2.5 and (24), we get the proof of Theorem 1.2. □

Proof of the Theorem 1.3 By Theorem 1.1 and Lemma 2.5, we get the proof of Theorem 1.3. □

References

1. S. S. Chern, *Finsler geometry is just Riemannian geometry without the quadratic equation*, Notices of the American Mathematical Society. 43.9 (1996), 959–963.
2. B. Chen, Z. Shen, and L. Zhao, *Constructions of Einstein Finsler metrics by warped product*, International Journal of Mathematics. 29.11 (2018), 1850081.
3. M. Gabrani, B. Rezaei, E. S. Sevim, *On Landsberg Warped Product Metrics*, Journal of Mathematical Physics, Analysis, Geometry 17. 4 (2021), 468–483.
4. L. Kozma, Ioan Radu Peter, and Csaba Varga. *Warped product of Finsler manifolds*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math 44 (2001), 157–170.
5. H. Liu, X. Mo, *Finsler warped product metrics of Douglas type*, Canadian Mathematical Bulletin. 62.1 (2019), 119–130.
6. H. Liu, X. Mo, H. Zhang, *Finsler warped product metrics with special Riemannian curvature properties*, Science China Mathematics 63.7 (2020), 1391–1408.