On the dynamics of lattice networks

Ghazaleh Malekbala*, Leila Sharifan

Hakim Sabzevari University, Sabzevar, Iran

Abstract. Let $(L, \leq)$ be a finite lattice and $f : L^n \to L^n$ be a lattice network over dependency graph $G = (V, E)$. We prove that f is a fixed-point system. Also we analyse fixed points of f is some special cases.

Keywords: Lattice network , Fixed points, Periodic points, Dependency graph

AMS Mathematics Subject Classification [2020]: 94C11, 94C15, 54H25)

---

1. Introduction

Finite dynamical systems play a crucial role in studying problems of several different contexts. In particular, these mathematical models, which naturally arise in computer processes, are profusely used in other sciences as biology, mathematics, physics, chemistry, or even sociology. Some relevant examples of finite dynamical systems are (finite) cellular automata and, more generally, deterministic Boolean networks, also called Boolean finite dynamical systems, diffusion models and recently semilattice networks (see [4, 5]. It is worth remarking that semilattice networks are generalization of conjunctive Boolean networks in [3] and some diffusion models that studied in [2] and the results of [5] recovers and extends some main theorems of those researches.

The benefit of finite dynamical systems is that they can be easily simulated on a computer in most cases and the difficulty is that few analytical devices beyond simulation are available. A common combinatorial device to study dynamics of models is the *dependency graph* that extensively has been used in Boolean networks, diffusion models and recently in the study of semilattice networks. Actually, [5] provides a general mathematical (algebraic and combinatorial) method to study semilattice networks.

A partially ordered set $(L, \leq)$ is called a join-semilattice if each two-element subset $\{a, b\} \subseteq L$ has a join (i.e. least upper bound, denoted by $a \vee b$), and is called a meet-semilattice if each two-element subset has a meet (i.e. greatest lower bound, denoted by $a \wedge b$). $(L, \leq)$ is called a lattice if it is both a join- and a meet-semilattice.

Let $(L, \leq)$ be a finite meet-semilattice (or join-semilattice). A finite dynamical system

$$f = (f_1, \ldots, f_n) : L^n \to L^n,$$

*Speaker. Email address: gmalekbala@gmail.com, is called a semilattice network when $f_j = \bigwedge_{x_i \in I_j} x_i$ for all $j = 1, \ldots, n$ where I_j is the set of variables that influence the variable x_j (or $f_j = \bigvee_{x_i \in I_j} x_i$ for all $j = 1, \ldots, n$). In this paper, we introduce and study the concept of lattice network. Let $(L, \leq)$ be a finite lattice. We say that a finite dynamical system

$$f = (f_1, \ldots, f_n) : L^n \to L^n,$$

is a lattice network when for each $j = 1, \ldots, n$, $f_j = \bigwedge_{x_i \in I_j} x_i$ or $f_j = \bigvee_{x_i \in I_j} x_i$. Note that the notion of lattice network is a generalization of the notion of AND-OR network where $L = \{0, 1\}$, $\wedge = AND$, and $\vee = OR$. Also note that AND-OR networks have been studied well in the literature. In the following, we assume that f is a lattice network. The dynamics of f is characterized by its *phase space* which is the directed graph with vertex set L^n and each $(\mathbf{u}, \mathbf{v})$ is a directed edge if $f(\mathbf{u}) = \mathbf{v}$. A *limit cycle* of length t is a set with t elements $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_t\}$ such that $f(\mathbf{u}_i) = \mathbf{u}_{i+1}$ for $i = 1, \ldots, t-1$ and $f(\mathbf{u}_t) = \mathbf{u}_1$. All elements of that limit cycle are called *periodic points* of period t. If $t = 1$ then $\mathbf{u}_1$ is called a fixed point and if all periodic points of f are fixed points then f is called a fixed-point network (system). Note that since L is a finite set, each $\mathbf{u} \in L^n$ converges to a periodic point (there exists $k \geq 0$ such that $f^k(\mathbf{u})$ is a periodic point). Dependency graph associated to f is a directed graph $G = (V, E)$ on the vertex set $V = \{1, 2, \ldots, n\}$ with the edge set

$$E = \{(i, j) \mid \text{ the function } f_j \text{ depends on } x_i\}$$

which is a powerful combinatorial tool that is used for detecting cycle structure of the network without direct computation of the phase space. In this paper, we assume that for each $1 \leq i \leq n$, $(i, i) \in E$ which means that each function f_i depends on x_i (Note that in many research papers this assumption has a crucial role in the study of the network), and we show that f is a fixed-point network.

2. Main results

Let $(L, \leq)$ be a finite lattice and $f : L^n \to L^n$ be a lattice network. In this section we show that f is a fixed-point system. Before presenting the main result let us give an example of lattice network.

Example 2.1. Let $L = \{1, 2, 3, 6\}$. Define $\leq$ in L by

$$a \leq b \quad \text{if and only if} \quad a \quad \text{divides} \quad b.$$

So, by $a \wedge b$ we mean the greatest common divisor of a and b, and by $a \vee b$ we mean the least common multiple of a and b.

Now define $f : L^3 \to L^3$ by $f(x_1, x_2, x_3) = (x_1 \vee x_2, x_2 \wedge x_3, x_3 \vee x_1)$. Then the dependency graph of f is $G = (V, E)$ where $V = \{1, 2, 3\}$ and $E = \{(1, 1), (2, 1), (2, 2), (3, 2), (3, 3), (1, 3)\}$. One can check that f is a fixed-point system and

$$fix(f) = \{(a, b, a) \in L^3 \mid b \mid a \text{ and } a \mid c\}$$

Now we are ready to present our main result.

Theorem 2.2. *Let $(L, \leq)$ be a finite lattice and $f : L^n \to L^n$ be a lattice network. Then f is a fixed point system.*

Proof. Let $G = (V, E)$ be the dependency graph of f. Since f is a lattice network, there exists $I, J \subseteq \{1, \ldots, n\}$ ($I \cup J = \{1, \ldots, n\}$, $I \cap J = \emptyset$) such that for each $j \in I$, $f_j = x_j \wedge \bigwedge_{(s,j) \in E} x_s$ and for each $j \in J$, $f_j = x_j \vee \bigvee_{(s,j) \in E} x_s$. Let $\mathbf{a} = (a_1^0, \ldots, a_n^0)$ be a t-periodic point of f. We show that $\mathbf{a}$ is a fixed point of f. For each $1 \leq r \leq t$ let $f^r(\mathbf{a}) = (a_1^r, \ldots, a_n^r)$ and let $L_1 := \{a_i^r \mid 1 \leq i \leq n, 0 \leq r \leq t\}$. For each $b \in L_1$ define $A_{r,b}$ $(0 \leq r \leq t)$ as

$$A_{r,b} = \{j \in I \mid a_j^r = b\}$$

Since $\mathbf{a}$ is a $t-$periodic point, it is clear that $A_{0,b} = A_{t,b}$ for all $b \in L_1$.

Suppose that b is a minimal element of L_1. Let $0 \leq r \leq t-1$ be such that $j \in A_{r,b}$, then by $f_j = x_j \wedge \bigwedge_{(s,j)\in E} x_s$, we have $a_j^{r+1} = a_j^r \wedge \bigwedge_{(s,j)\in E} a_s^r$. So $a_j^{r+1} \leq a_j^r = b$ and by the minimality of b in L_1, We get $a_j^{r+1} = b$ which shows that $j \in A_{r+1,b}$ and so for each $0 \leq r \leq t-1$, $A_{r,b} \subseteq A_{r+1,b}$.mThus

$$A_{0,b} \subseteq A_{1,b} \subseteq \cdots \subseteq A_{r,b} = A_{0,b}$$

In particular,

$$A_{0,b} = A_{1,b}$$

Now suppose that $b \in L_1$ is a cover of some minimal element of L_1. Let $0 \leq r \leq t-1$ be such that $j \in A_{r,b}$. As previous paragraph, we can show that $a_j^{r+1} \leq a_j^r = b$. If $a_j^{r+1} < a_j^r$, then a_j^{r+1} is a minimal element of L_1 and so $A_{r,a_j^{r+1}} = A_{r+1,a_j^{r+1}}$ and it shows that $a_j^r = a_j^{r+1}$ which is a contradiction. So $a_j^{r+1} = a_j^r$ and again we conclude that for each $0 \leq r \leq t-1$, $A_{r,b} \subseteq A_{r+1,b}$. So,

$$A_{0,b} \subseteq A_{1,b} \subseteq \cdots \subseteq A_{r,b} = A_{0,b},$$

and we have

$$A_{0,b} = A_{1,b}.$$

Continuing this process, we finally conclude that

$$\forall b \in L_1,\ A_{0,b} = A_{1,b} \tag{1}$$

Now for each $b \in L_1$ define $B_{r,b}$ $(0 \leq r \leq t)$ as

$$B_{r,b} = \{j \in J \mid a_j^r = b\}$$

Since $\mathbf{a}$ is a $t-$periodic point, it is clear that $B_{0,b} = B_{t,b}$ for all $b \in L_1$. Suppose that b is a maximal element of L_1. Let $0 \leq r \leq t-1$ be such that $j \in B_{r,b}$, then by $f_j = x_j \vee \bigvee_{(s,j)\in E} x_s$, we have $a_j^{r+1} = a_j^r \vee \bigvee_{(s,j)\in E} a_s^r$. So $b = a_j^r \leq a_j^{r+1}$ and by the maximality of b in L_1, We get $a_j^{r+1} = b$ which shows that $j \in A_{r+1,b}$ and as above this yields to the fact that

$$A_{0,b} = A_{1,b}$$

Next we do the same method for an arbitrary element of L_1 which is covered by a maximal element of L_1 and continuing in this way we get that

$$\forall b \in L_1,\ A_{0,b} = A_{1,b} \tag{2}$$

Now by equations (1) and (2), $\mathbf{a}$ is a fixed point of f. □

We remark that Theorem 2.2 is an extension of [**?**, Theorem 1] where f is finite Boolean dynamical system and $\wedge, \vee$ are AND, OR respectively. Next we study some special cases.

Corollary 2.3. *Let $(L, \leq)$ be a finite lattice and $f : L^n \to L^n$ be a lattice network with dependency graph $G = (V, E)$. Then the following holds.*

(i) *If for each* $1 \leq j \leq n$, $f_j = \bigwedge_{(s,j)\in E} x_s$, *then an arbitrary point* $\mathbf{a} = (a_1, \ldots, a_n)$ *converges to the fixed point* $\mathbf{b_a} = (b_1, \ldots, b_n)$ *where for each* $1 \leq i \leq n$,

$$b_i = \bigwedge_{\text{there is a directed path from } s \text{ to } i} a_s$$

(ii) *If for each* $1 \leq j \leq n$, $f_j = \bigvee_{(s,j)\in E} x_s$, *then an arbitrary point* $\mathbf{a} = (a_1, \ldots, a_n)$ *converges to the fixed point* $\mathbf{b_a} = (b_1, \ldots, b_n)$ *where for each* $1 \leq i \leq n$,

$$b_i = \bigvee_{\text{there is a directed path from } s \text{ to } i} a_s$$

(iii) *If* f *is as (i) or (ii), and* G *is strongly connected then an arbitrary point* $\mathbf{a} = (a_1, \ldots, a_n)$ *converges to the fixed point* $\mathbf{b_a} = (b_1, \ldots, b_n)$ *where for each* $1 \leq i \leq n$, $b_i = \bigvee_{1\leq s\leq n} a_s$ *and in particular* $\mathbf{a} = (a_1, \ldots, a_n)$ *is a fixed point of* f *if and only if* $a_1 = \cdots = a_n$.

3. Conclusion

Let f be a lattice network over a dependency graph $G = (V, E)$. We prove that f is a fixed-point system provided that all vertices of G have a self-loop.

References

1. Aledo, JA; Martinez, M; Valverde, JC. Parallel discrete dynamical systems on independent local functions Journal of Computational and Applied Mathematics 237 (1), 335–339
2. Gao, Z; Chen, X; Liu, T; Baar, T. Periodic behavior of a diffusion model over directed graphs, In 2016 IEEE 55th conference on decision and control (CDC). (2016) 37–42.
3. Jarrah, A. S; Laubenbacher, R; Veliz-Cuba, A. The dynamics of conjunctive and disjunctive Boolean network models, Bull. Math. Bio. 72 (2010) 1425–1447.
4. Malekbala, G; Musavizadeh Jazaeri, L; Sharifan, L; Taha, M. On the dynamics of semilattice networks, Submitted.
5. Veliz-Cuba, A; Laubenbacher, R. Dynamics of semilattice networks with strongly connected dependency graph, Automatica. 99 (2019), 167–174.