On generalized symmetric Finsler spaces with Matsumoto metrics

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Abstract. In this paper, we study generalized symmetric $(\alpha, \beta)-$spaces. We prove that generalized symmetric $(\alpha, \beta)-$spaces with Matsumoto metric are Riemannian.

Keywords: $(\alpha, \beta)-$metric, generalized symmetric space, Matsumoto metric

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1. Introduction

The notion of symmetric spaces is due to Cartan. Later, Kowalski [3] defined generalized symmetric spaces or regular $s-$spaces following the introduction of $s-$manifolds in [?]. Generalized symmetric Finsler spaces are a natural generalization of generalized symmetric spaces and they keep many of their properties [2, 6] . Let (M, F) be a connected Finsler manifold. A symmetry at $x \in M$ is an isometry of (M, F) for which x is an isolated fixed point. A $s-$structure on (M, F) is a family $\{s_x\}_{x\in M}$ such that s_x is a symmetry at $x \in M$, for each $x \in M$. An $s-$structure is called regular if for any two points $x, y \in M$

$$s_x \circ s_y = s_z \circ s_x, \qquad z = s_x(y).$$

An $s-$structure $\{s_x\}_{x\in M}$ is called of order k if $(s_x)^k = id_M$ for all $x \in M$ and k is the minimal number with this property. It is well known that if (M, F) admits an $s-$structure, then it always admits an $s-$structure of finite order. In particular if (M, F) admits an $s-$structure of order two then it is a usual symmetric Finsler space.

An $(\alpha, \beta)-$metric is a Finsler metric of the form $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ where $\alpha = \sqrt{\tilde{a}_{ij}(x)y^iy^j}$ is induced by a Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ on a connected smooth $n-$dimensional manifold M and $\beta = b_i(x)y^i$ is a $1-$form on M. Some important classes of $(\alpha, \beta)-$metrics are Randers metric $F = \alpha + \beta$, Matsumoto metric $F = \frac{\alpha^2}{(\alpha-\beta)}$, infinite series metic $F = \frac{\beta^2}{\beta-\alpha}$ and exponential metric $F = \alpha \exp(\frac{\beta}{\alpha})$

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2. Preliminaries

Let M be a smooth $n-$dimensional C^∞ manifold and TM be its tangent bundle. A Finsler metric on a manifold M is a non-negative function $F : TM \longrightarrow R$ with the following properties [1]:

(1) F is smooth on the slit tangent bundle $TM^0 := TM \setminus \{0\}$.
(2) $F(x, \lambda y) = \lambda F(x, y)$ for any $x \in M$, $y \in T_xM$ and $\lambda > 0$.
(3) The following bilinear symmetric form $g_y : T_xM \times T_xM \longrightarrow R$ is positive definite

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t} F^2(x, y + su + tv)|_{s=t=0}.$$

Definition 2.1. Let $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ be a norm iduced ba a Riemannian metric $\tilde{a}$ and $\beta(x, y) = b_i(x)y^i$ be a $1-$form on an $n-$dimensional manifold M. Let

$$\|\beta(x)\|_\alpha := \sqrt{\tilde{a}^{ij}(x)b_i(x)b_j(x)}. \tag{2.1}$$

Now, let the function F is defined as follows

$$F := \alpha\phi(s) \quad , \quad s = \frac{\beta}{\alpha}, \tag{2.2}$$

where $\phi = \phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0 \quad , \quad |s| \le b < b_0. \tag{2.3}$$

Then F is a Finsler metric if $\|\beta(x)\|_\alpha < b_0$ for any $x \in M$. A Finsler metric in the form (2.2) is called an $(\alpha, \beta)-$metric.

A Finsler space having the Finsler function:

$$F(x, y) = \frac{\alpha^2(x, y)}{\alpha(x, y) - \beta(x, y)} \tag{2.4}$$

is called a Matsumoto space.

The Riemannian metric $\tilde{a}$ induces an inner product on any cotangent space T_x^*M such that $\langle dx^i(x), dx^j(x)\rangle = \tilde{a}^{ij}(x)$. The induced inner product on T_x^*M induces a linear isomorphism between T_x^*M and T_xM. Then the 1-form β corresponds to a vector field $\tilde{X}$ on M such that

$$\tilde{a}(y, \tilde{X}(x)) = \beta(x, y). \tag{2.5}$$

Also we have $\|\beta(x)\|_\alpha = \|\tilde{X}(x)\|_\alpha$. Therefore we can write $(\alpha, \beta)-$metrics as follows:

$$F(x, y) = \alpha(x, y)\phi(\frac{\tilde{a}(\tilde{X}(x), y)}{\alpha(x, y)}), \tag{2.6}$$

where for any $x \in M$, $\sqrt{\tilde{a}(\tilde{X}(x), \tilde{X}(x))} = \|\tilde{X}(x)\|_\alpha < b_0$. Symmetric Finsler spaces form a natural extension to the symmetric spaces of Cartan. A symmetric Finsler spaces is a Finsler space (M, F) such that for all $p \in M$ there exist an involutive isometry $s_p \in M$ such that p is an isolated fixed point of s_p [4]. Generalized symmetric Finsler spaces were introduced as generalization of generalized symmetric spaces [2]. A Finsler space (M, F) is said to be symmetric space if for any point $p \in M$ there exist an involutive isometry s_p of (M, F) such that p is an isolated fixed point of (M, F) . Let (M, F) be a connected Finsler space. An isometry s_x of (M, F) for which $x \in M$ is an isolated fixed point will be called a symmetry of M at x.

An $s-$structure on (M,F) is a family $\{s_x|x\in M\}$ of symmetries of (M,F). The corresponding tensor field S of type (1,1) defined by $S_x=(s_x)_x$ for each $x\in M$ is called the symmetry tensor field of $s-$structure [2, 3].

Definition 2.2. An $s-$structure $\{s_x|x\in M\}$ on a Finsler space (M,F) is said to be regular if it satisfies the rule

$$s_x\circ s_y=s_z\circ s_x,\qquad z=s_x(y)$$

for every two points $x,y\in M$.

3. Generalized symmetric (α,β) spaces

Lemma 3.1. Let (M,F) be a generalized symmetric Matsumoto space with F defined by the Riemannian metric $\tilde{a}$ and the vector field X. Then the regular $s-$structure $\{s_x\}$ of (M,F) is also a regular $s-$structure of the Riemannian manifold $(M,\tilde{a})$.

Proof: Let s_x be a symmetry of (M,F) at x and $p\in M$. Then for any $Y\in T_pM$ we have

$$\begin{aligned}
F(p,Y) &= F(s_x(p),ds_x(Y))\\
\frac{\tilde{a}(Y,Y)}{\sqrt{\tilde{a}(Y,Y)}-\tilde{a}(X_p,Y)} &= \frac{\tilde{a}(ds_xY,ds_xY)}{\sqrt{\tilde{a}(ds_xY,ds_xY)}-\tilde{a}(X_{s_x(p)},ds_xY)}.
\end{aligned}$$

Applying the above equation to $-Y$, we get

$$\frac{\tilde{a}(Y,Y)}{\sqrt{\tilde{a}(Y,Y)}+\tilde{a}(X_p,Y)}=\frac{\tilde{a}(ds_xY,ds_xY)}{\sqrt{\tilde{a}(ds_xY,ds_xY)}+\tilde{a}(X_{s_x(p)},ds_xY)}.$$

Combining the above two equations, we get

$$\begin{aligned}
\tilde{a}(Y,Y) &= \tilde{a}(ds_xY,ds_xY)\\
\tilde{a}(X_p,Y) &= \tilde{a}(X_{s_x(p)},ds_xY).
\end{aligned}$$

Thus s_x is a symmetry with respect to the Riemannian metric $\tilde{a}$.□

Lemma 3.2. Let $(M,\tilde{a})$ be a generalized symmetric Riemannian space. Also suppose that F is a Matsumoto metric introduced by $\tilde{a}$ and a vector field X. Then the regular $s-$structure $\{s_x\}$ of $(M,\tilde{a})$ is also a regular $s-$structure of (M,F) if and only if X is s_x-invariant for all $x\in M$.

Proof: Let X be s_x-invariant. Therefore for any $p\in M$, we have $X_{s_x(p)}=ds_xX_p$. Then for any $y\in T_pM$ we have

$$\begin{aligned}
F(s_x(p),ds_xy_p) &= \frac{\tilde{a}(ds_xy_p,ds_xy_p)}{\sqrt{\tilde{a}(ds_xy,ds_xy)}-\tilde{a}(X_{s_x(p)},ds_xy)}\\
&= \frac{\tilde{a}(y,y)}{\sqrt{\tilde{a}(y,y)}-\tilde{a}(ds_xX_p,ds_xy)}\\
&= \frac{\tilde{a}(y,y)}{\sqrt{\tilde{a}(y,y)}-\tilde{a}(X_p,y)}.\\
&= F(p,y)
\end{aligned}$$

Conversely, let s_x be a symmetry of (M,F) at x. Then for any $p\in M$ and $y\in T_pM$ we have

$$F(p,y)\ =\ F(s_x(p),ds_xy)$$

$$\frac{\tilde{a}(y,y)}{\sqrt{\tilde{a}(y,y)} - \tilde{a}(X_p,y)} \quad = \quad \frac{\tilde{a}(ds_x y_p, ds_x y_p)}{\sqrt{\tilde{a}(ds_x y, ds_x y)} - \tilde{a}(X_{s_x(p)}, ds_x y)}.$$

So we have

$$\tilde{a}(ds_x X_p - X_{s_x(p)}, ds_x y_p) = 0.$$

Therefore $ds_x X_p = X_{s_x(p)}$.□

THEOREM 3.3. *A generalized symmetric Matsumoto space must be Riemannian.*

Proof: Let (M,F) be a generalized symmetric Matsumoto space with F defined by the Riemannian metric $\tilde{a}$ and the vector field X, and let $\{s_x\}$ be the regular $s-$structure of (M,F). Let s_x be a symmetry of (M,F). Then by lemma 3.1, s_x is also a symmetry of $(M,\tilde{a})$. Thus we have

$$\begin{aligned} F(x, ds_x(y)) &= \frac{\tilde{a}(ds_x y, ds_x y)}{\sqrt{\tilde{a}(ds_x y, ds_x y)} - \tilde{a}(X_x, ds_x y)} \\ &= \frac{\tilde{a}(y,y)}{\sqrt{\tilde{a}(y,y)} - \tilde{a}(X_x, ds_x y)} \\ &= F(x,y). \end{aligned}$$

Therefore $\tilde{a}(X_x, ds_x y) = \tilde{a}(X_x, y)$, $\forall y \in T_xM$. Since x is an isolated fixed point of the symmetry s_x, the tangent map $S_x = (ds_x)_x$ is an orthogonal transformation of T_xM having no nonzero fixed vectors. So we have $\tilde{a}(X_x, (S - id)_x(y)) = 0$, $\forall y \in T_xM$. Since $(S - id)_x$ is an invertible linear transformation, we have $X_x = 0$, $\forall x \in M$. Hence F is Riemannian. □

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