Various shadowing properties for time varying maps

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Abstract. In this paper we study various notions of shadowing of dynamical systems so-called time varying maps. We define and study the h-shadowing, limit shadowing, s-limit shadowing and exponential limit shadowing properties of these dynamical systems. We investigate the relationships between these notions of shadowing and examine the role that expansivity plays in shadowing properties of such dynamical systems. Specially, we prove some results linking s-limit shadowing property to limit shadowing property, and h-shadowing property to s-limit shadowing and limit shadowing properties.

Keywords: Time varying map, h-shadowing, Limit shadowing, s-limit shadowing, Exponential limit shadowing.

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1. Introduction

The time varying maps (so-called non-autonomous or time-dependent dynamical systems), describe situations where the dynamics can vary with time and yield very flexible models than autonomous cases for the study and description of real world processes. They may be used to describe the evolution of a wider class of phenomena, including systems which are forced or driven. In the recent past, lots of studies have been done regarding dynamical properties in such systems, but a global theory is still out of reach. In general time varying maps can be rather complicated. Thus, we are inclined to look at approximations of orbits, also called pseudo orbits. Systems for which pseudo orbits can be approximated by true orbits are said to satisfy the shadowing property. The shadowing property plays a key role in the study of the stability of dynamical systems. This property is found in hyperbolic dynamics, and it was used to prove their stability. In this literature, some remarkable results were further obtained through works of several authors, see e.g. [1–3]. Since the approximation by true orbits can be expressed in various ways, different notions of shadowing have been introduced. In this paper, what we want to study on time varying maps is shadowing, h-shadowing, limit shadowing, s-limit shadowing and exponential limit shadowing properties.

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This article is organized as follows. In Section 2, we present an overview of the main concepts and introduce notations. Next, we give our main results in Section 3.

2. Preliminaries

Throughout this paper we consider (X, d) to be a metric space, $f_n : X \to X$, $n \in \mathbb{N}$, to be a sequence of continuous maps and $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ to be a time varying map on X that its time evolution is defined by composing the maps f_n in the following way

$$\mathcal{F}_n := f_n \circ f_{n-1} \circ \cdots \circ f_1, \text{ for } n \geq 1, \text{ and } \mathcal{F}_0 := Id_X.$$

For time varying map $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ defined on X, we set $\mathcal{F}_{[i,j]} := f_j \circ f_{j-1} \circ \cdots \circ f_{i+1} \circ f_i$ for $1 \leq i \leq j$, and $\mathcal{F}_{[i,j]} := Id_X$ for $i > j$. Also, for any $k > 0$, we define a time varying map (k^{th}-iterate of $\mathcal{F}$) $\mathcal{F}^k = \{g_n\}_{n\in\mathbb{N}}$ on X, where

$$g_n = f_{nk} \circ f_{(n-1)k+k-1} \circ \ldots \circ f_{(n-1)k+2} \circ f_{(n-1)k+1} \text{ for } n \geq 1.$$

Thus $\mathcal{F}^k = \{\mathcal{F}_{[(n-1)k+1,nk]}\}_{n\in\mathbb{N}}$.

Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a time varying map on a metric space (X, d). For a point $x_0 \in X$, put $x_n := \mathcal{F}_n(x_0)$ for all $n \geq 0$. Then the sequence $\{x_n\}_{n\geq 0}$, denoted by $\mathcal{O}(x_0)$, is said to be the *orbit* of x_0 under time varying map $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$. Moreover, a subset Y of X is said to be *invariant* under $\mathcal{F}$ if $f_n(Y) = Y$ for all $n \geq 1$, equivalently $\mathcal{F}_n(Y) = Y$ for all $n \geq 0$.

Definition 2.1 (Conjugacy). Let (X, d_1) and (Y, d_2) be two metric spaces. Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ and $\mathcal{G} = \{g_n\}_{n\in\mathbb{N}}$ be time varying maps on X and Y, respectively. If there exists a homeomorphism $h : X \to Y$ such that $h \circ f_n = g_n \circ h$, for all $n \in \mathbb{N}$, then $\mathcal{F}$ and $\mathcal{G}$ are said to be *conjugate* (with respect to the map h) or *h-conjugate*. In particular, if $h : X \to Y$ is a uniform homeomorphism, then $\mathcal{F}$ and $\mathcal{G}$ are said to be *uniformly conjugate* or *uniformly h-conjugate*. (Recall that homeomorphism $h : X \to Y$, such that h and h^{-1} are uniformly continuous, is called a uniform homeomorphism.)

Definition 2.2 (Shadowing property). Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a time varying map on a metric space (X, d) and Y be a subset of X. Then,

(1) for $\delta > 0$, a sequence $\{x_n\}_{n\geq 0}$ in X is said to be a *δ-pseudo orbit* if

$$d(f_{n+1}(x_n), x_{n+1}) < \delta \text{ for all } n \geq 0;$$

(2) for given $\varepsilon > 0$, a δ-pseudo orbit $\{x_n\}_{n\geq 0}$ is said to be *ε-shadowed* by $x \in X$ if $d(\mathcal{F}_n(x), x_n) < \varepsilon$ for all $n \geq 0$;

(3) the time varying map $\mathcal{F}$ is said to have *shadowing property* on Y if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that every δ-pseudo orbit in Y is ε-shadowed by some point of X. If this property holds on $Y = X$, we simply say that $\mathcal{F}$ has *shadowing property*.

Definition 2.3 (h-shadowing property). Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a time varying map on a metric space (X, d) and Y be a subset of X. We say that $\mathcal{F}$ has *h-shadowing property* on Y if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every finite δ-pseudo orbit $\{x_0, x_1, \ldots, x_m\}$ in Y there is $x \in X$ such that $d(\mathcal{F}_n(x), x_n) < \varepsilon$ for every $0 \leq n < m$ and $\mathcal{F}_m(x) = x_m$. If this property holds on $Y = X$, we simply say that $\mathcal{F}$ has *h-shadowing property*.

Definition 2.4 (Limit shadowing property). Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a time varying map on a metric space (X, d) and Y be a subset of X. Then,

(1) a sequence $\{x_n\}_{n\geq 0}$ in X is called a *limit pseudo orbit* if $d(f_{n+1}(x_n), x_{n+1}) \to 0$ as $n \to +\infty$;
(2) a sequence $\{x_n\}_{n\geq 0}$ in X is said to be *limit shadowed* if there is $x \in X$ such that $d(\mathcal{F}_n(x), x_n) \to 0$, as $n \to +\infty$;
(3) the time varying map $\mathcal{F}$ has the *limit shadowing property* on Y whenever every limit pseudo orbit in Y is limit shadowed by some point of X. If this property holds on $Y = X$, we simply say that $\mathcal{F}$ has *limit shadowing property.*

The notion of limit shadowing property was extended to a notion so called s-limit shadowing property, to account the fact that many systems have limit shadowing property but not shadowing property.

Definition 2.5 (s-Limit shadowing property). Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a time varying map on a metric space (X, d) and Y be a subset of X. We say that $\mathcal{F}$ has *s-limit shadowing property* on Y if for every $\varepsilon > 0$ there is $\delta > 0$ such that

(1) for every δ-pseudo orbit $\{x_n\}_{n\geq 0}$ in Y, there exists $x \in X$ satisfying $d(\mathcal{F}_n(x), x_n) < \varepsilon$ for all $n \geq 0$, and,
(2) if in addition, $\{x_n\}_{n\geq 0}$ is a limit pseudo orbit in Y then $d(\mathcal{F}_n(x), x_n) \to 0$ as $n \to +\infty$.

If this property holds on $Y = X$, we simply say that $\mathcal{F}$ has *s-limit shadowing property.*

We say that a sequence $\{a_n\}_{n\geq 0}$ of real numbers converges to zero with rate $\theta \in (0, 1)$ and write $a_n \xrightarrow{\theta} 0$ as $n \to +\infty$, if there exists a constant $L > 0$ such that $|a_n| \leq L\theta^n$ for all $n \geq 0$.

Definition 2.6 (Exponential limit shadowing property). Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a time varying map on a metric space (X, d) and Y be a subset of X. Then,

(1) for $\theta \in (0, 1)$, a sequence $\{x_n\}_{n\geq 0}$ in X is called a *θ-exponentially limit pseudo orbit* of $\mathcal{F}$ if $d(f_{n+1}(x_n), x_{n+1}) \xrightarrow{\theta} 0$ as $n \to +\infty$;
(2) the time varying map $\mathcal{F}$ has the *exponential limit shadowing property with exponent* ξ on Y if there exists $\theta_0 \in (0, 1)$ so that for any θ-exponentially limit pseudo orbit $\{x_n\}_{n\geq 0} \subseteq Y$ with $\theta \in (\theta_0, 1)$, there is $x \in X$ such that $d(\mathcal{F}_n(x), x_n) \xrightarrow{\theta^\xi} 0$, as $n \to +\infty$. In the case $\xi = 1$ we say that F has the *exponential limit shadowing property* on Y. If this property holds on $Y = X$, we simply say that $\mathcal{F}$ has *exponential limit shadowing property.*

3. Main results

In this section, we mention our main results.

Theorem 3.1. *Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ and $\mathcal{G} = \{g_n\}_{n\in\mathbb{N}}$ be time varying maps on metric spaces (X, d_1) and (Y, d_2), respectively, such that $\mathcal{F}$ is uniformly conjugate to $\mathcal{G}$. Then, the following statements hold:*

(a) *If $\mathcal{F}$ has the h-shadowing property, then so does $\mathcal{G}$.*
(b) *If $\mathcal{F}$ has the limit shadowing property, then so does $\mathcal{G}$.*
(c) *If $\mathcal{F}$ has the s-limit shadowing property, then so does $\mathcal{G}$.*

Theorem 3.2. *Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ and $\mathcal{G} = \{g_n\}_{n\in\mathbb{N}}$ be time varying maps on metric spaces (X, d_1) and (Y, d_2), respectively. Define metric d on $X \times Y$ by*

$$d((x_1, y_1), (x_2, y_2)) := \max\{d_1(x_1, x_2), d_2(y_1, y_2)\} \quad \textit{for any } (x_1, y_1), (x_2, y_2) \in X \times Y.$$

Then,

(a) $\mathcal{F}$ *and* $\mathcal{G}$ *have the h-shadowing property if and only if so does* $\mathcal{F} \times \mathcal{G} := \{f_n \times g_n\}_{n\in\mathbb{N}}$*.*
(b) $\mathcal{F}$ *and* $\mathcal{G}$ *have the limit shadowing property if and only if so does* $\mathcal{F} \times \mathcal{G}$*.*
(c) $\mathcal{F}$ *and* $\mathcal{G}$ *have the exponential limit shadowing property if and only if so does* $\mathcal{F} \times \mathcal{G}$*.*
(d) $\mathcal{F}$ *and* $\mathcal{G}$ *have the s-limit shadowing property if and only if so does* $\mathcal{F} \times \mathcal{G}$*.*

THEOREM 3.3. *Let* $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ *be a time varying map on metric space* (X, d) *and* $k \in \mathbb{N}$*. Then, the following statements hold:*

(a) *If* $\mathcal{F}$ *has the limit shadowing property, then so does* $\mathcal{F}^k$*.*
(b) *If* $\mathcal{F}$ *has the exponential limit shadowing property, then so does* $\mathcal{F}^k$*.*
(c) *If* $\mathcal{F}$ *has the s-limit shadowing property, then so does* $\mathcal{F}^k$*.*

DEFINITION 3.4 (Equicontinuity). Time varying map $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ on a metric space (X, d) is said to be *equicontinuous* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d(\mathcal{F}_{[i,j]}(x), \mathcal{F}_{[i,j]}(y)) < \varepsilon$ for all $1 \le i \le j$.

THEOREM 3.5. *Let* $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ *be an equicontinuous time varying map on a compact metric space* (X, d) *and* Y *be an invariant subset of* X*. Then, the following conditions are equivalent:*

(a) $\mathcal{F}$ *has the h-shadowing property on* Y*.*
(b) $\mathcal{F}^k$ *has the h-shadowing property on* Y *for some* $k \in \mathbb{N}$*.*
(c) $\mathcal{F}^k$ *has the h-shadowing property on* Y *for all* $k \in \mathbb{N}$*.*

LEMMA 3.6. *Let* $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ *be a time varying map on a metric space* (X, d) *and* Y *be a subset of* X*. If* $Y \subseteq f_n(Y)$ *for every* $n \in \mathbb{N}$ *and* $\mathcal{F}$ *has s-limit shadowing property on* Y *then* $\mathcal{F}$ *also has limit shadowing property on* Y*. In particular, if* $\mathcal{F}$ *is a time varying map of surjective maps and has s-limit shadowing property then* $\mathcal{F}$ *also has limit shadowing property.*

THEOREM 3.7. *Let* $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ *be a time varying map on a compact metric space* (X, d) *and* Y *be a closed subset of* X*. Then, the following statements hold:*

(a) *If there is an open set* U *such that* $Y \subseteq U$ *and* $\mathcal{F}$ *has h-shadowing property on* U*, then* $\mathcal{F}$ *has s-limit shadowing property on* Y*. If in addition,* $Y \subseteq f_n(Y)$ *for every* $n \in \mathbb{N}$ *then* $\mathcal{F}$ *has limit shadowing property on* Y*.*
(b) *If* Y *is invariant and* $\mathcal{F}|_Y$ *has h-shadowing property then* $\mathcal{F}|_Y$ *has s-limit shadowing property and limit shadowing property.*
(c) *If* $\mathcal{F}$ *has h-shadowing property then* $\mathcal{F}$ *has s-limit shadowing property. If in addition,* $\mathcal{F}$ *is a time varying map of surjective maps then* $\mathcal{F}$ *has limit shadowing property.*

DEFINITION 3.8 (Expansivity). An time varying map $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ on a metric space (X, d) is called *strongly expansive* if there exists $\gamma > 0$ (called expansivity constant) such that for any two distinct points $x, y \in X$ and every $N \in \mathbb{N}$, $d(\mathcal{F}_{[N,n]}(x), \mathcal{F}_{[N,n]}(y)) > \gamma$ for some $n \ge N$. Equivalently, if for $x, y \in X$ and some $N \in \mathbb{N}$, $d(\mathcal{F}_{[N,n]}(x), \mathcal{F}_{[N,n]}(y)) \le \gamma$ for all $n \ge N$, then $x = y$.

COROLLARY 3.9. *Let* $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ *be a time varying map on a compact metric space* (X, d)*.*

(a) *If $\mathcal{F}$ is strongly expansive then $\mathcal{F}$ has the shadowing property if and only if $\mathcal{F}$ has the h-shadowing property.*
(b) *If $\mathcal{F}$ is strongly expansive and has the shadowing property then $\mathcal{F}$ has the h-shadowing and s-limit shadowing properties. If in addition, $\mathcal{F}$ is a time varying map of surjective maps then $\mathcal{F}$ has the limit shadowing property.*

DEFINITION 3.10 (Uniformly contracting and uniformly expanding time varying map). Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a time varying map on a metric space (X, d). Then,

(1) the time varying map $\mathcal{F}$ is *uniformly contracting* if its contracting ratio which denoted by α exists and is less than one, where

$$\alpha := \sup_{n\in\mathbb{N}} \sup_{\substack{x,y\in X\\ x\neq y}} \frac{d(f_n(x), f_n(y))}{d(x, y)};$$

(2) the time varying map $\mathcal{F}$ is *uniformly expanding* if its expanding ratio which denoted by β exists and is greater than one, where

$$\beta := \inf_{n\in\mathbb{N}} \inf_{\substack{x,y\in X\\ x\neq y}} \frac{d(f_n(x), f_n(y))}{d(x, y)}.$$

THEOREM 3.11. *Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a uniformly contracting time varying map on a metric space (X, d). Then, $\mathcal{F}$ has the shadowing, limit shadowing, s-limit shadowing and exponential limit shadowing properties.*

THEOREM 3.12. *Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a uniformly expanding time varying map of surjective maps on a complete metric space (X, d). Then, $\mathcal{F}$ has the shadowing, limit shadowing, s-limit shadowing and exponential limit shadowing properties.*

DEFINITION 3.13. Let $f : X \to X$ be a linear homeomorphism on a Banach space X. Then, f is said to be *hyperbolic* if there exist Banach subspaces $X_s, X_u \subset X$, called stable and unstable subspaces, respectively, and a norm $\|.\|$ on X compatible with the original Banach structure such that

$$X = X_s \oplus X_u, \quad f(X_s) = X_s, \quad f(X_u) = X_u, \quad \|f|_{X_s}\| < 1 \ \text{ and } \ \|f^{-1}|_{X_u}\| < 1.$$

THEOREM 3.14. *Let X be a Banach space, and let $\mathcal{A}$ be a finite set of hyperbolic linear homeomorphisms with the same stable and unstable subspaces. Then, any time varying map $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ with $f_n \in \mathcal{A}$ has the shadowing, limit shadowing, s-limit shadowing and exponential limit shadowing properties.*

4. Conclusion

Time varying maps which are a natural generalization of autonomous dynamical systems, are more flexible tools for the description and study of real world processes. Hence, their study is important.

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